Name: David Fordham

Problem	Points	Score
1a	10	
1b	10	
2a	10	
2b	10	
2c	10	
3a	10	
3b	10	
3c	10	
3d	10	
3e	10	
Total	100	

Notes:

1. The exam is closed books/closed notes - except for one page of notes.

2. Please show ALL work. Incorrect answers with no supporting explanations or work will be given no partial credit.

3. Please indicate clearly your answer to the problem. If I can't read it (and I am the judge of legibility), it is wrong. If I can't follow your solution (and I get lost easily), it is wrong. All things being equal, neat and legible work will get the higher grade:)

Problem No. 1: Show the following properties of the Fourier transform for **real** signals:

(a) $|X(f)| = |X(-f)|$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$$
$$= \int_{-\infty}^{\infty} x(t)\cos(2\pi ft)dt - j\int_{-\infty}^{\infty} x(t)\sin(2\pi ft)dt$$
$$= \int_{-\infty}^{\infty} x(t)\cos(2\pi ft)dt$$

To prove this we first start with the definition of the Fourier Transform.
Next, Eulers's Theorem is applied to the integral.
Since the magnitude is an even function of frequency and the frequency dependence of the real part of X(f) is through cos2πft, it is therefore an even function of f. Therefore, by taking the magnitude and integrating the above function the following derivation occurs:

$$X(f) = \int_{-\infty}^{\infty} x(-t)\cos(-2\pi ft)dt = \int_{-\infty}^{\infty} x(-t)\cos(2\pi ft)dt$$

In conclusion, we know cos(t) = cos(-t) which means it is an even function. So, from the above derivations we see that:

X(f) = X(-f)

(b) $\theta(f) = -\theta(-f)$

$$\theta(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$$
$$= \int_{-\infty}^{\infty} x(t)\cos(2\pi ft)dt - j\int_{-\infty}^{\infty} x(t)\sin(2\pi ft)dt$$
$$= -j\int_{-\infty}^{\infty} x(t)\sin(2\pi ft)dt$$

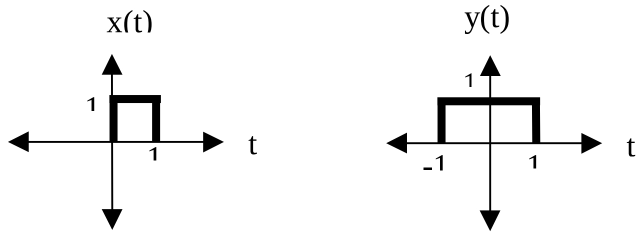
To prove this we first start with the definition of the Fourier Transform.
Next, Eulers's Theorem is applied to the integral.
Since the frequency dependence of the imaginary part of X(f) is through sin2πft, it is therefore an odd function of f. The following derivations prove this point.

$$\theta(-f) = -j\int_{-\infty}^{\infty} x(-t)\sin 2\pi f(-t)dt$$
$$\theta(-f) = j\int_{-\infty}^{\infty} x(-t)\sin 2\pi ftdt$$

In conclusion, we know that sin(t) = -sin(-t) which means it is an odd function. So, from the above derivations we see that

θ(f) = -θ(-f)

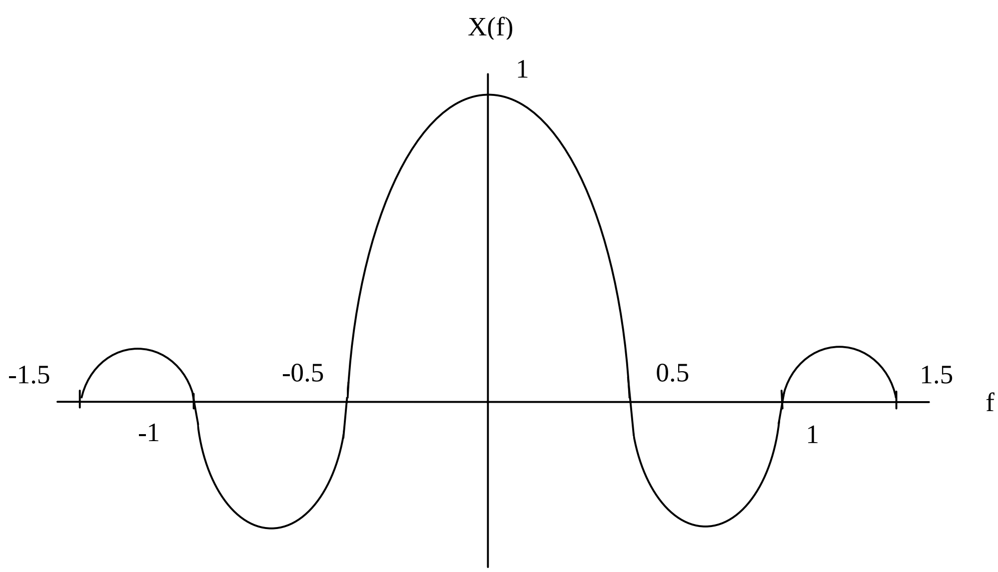
Problem No. 2: For the following two signals:



(a) Plot the magnitude spectrum of *x*(*t*):

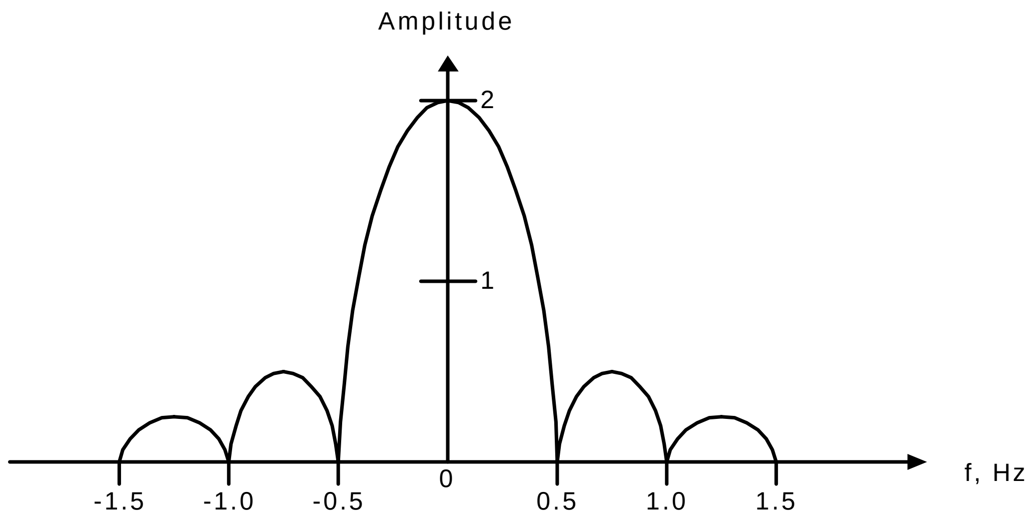
Using MathCad to evaluate this function and graph the output, the following was attained.

$$X(f) = \int_0^1 e^{-j2\pi ft}\,dt = -\left.\frac{e^{-j2\pi ft}}{j2\pi f}\right|_0^1 = -\frac{1}{j2\pi f}\left[e^{-j2\pi f} - 1\right]$$



(b) Plot the magnitude spectrum of $y(t)$:

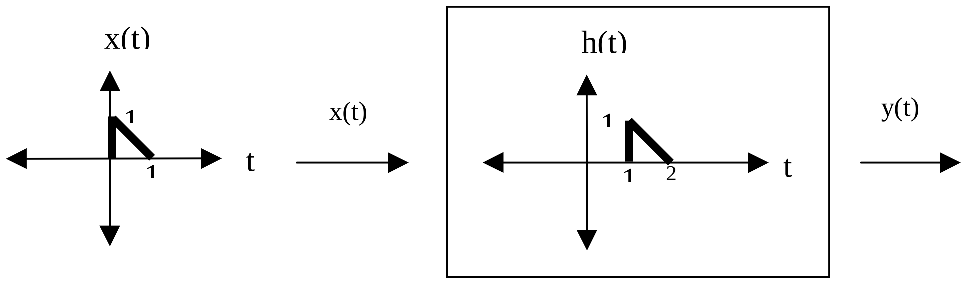
$$X(f) = \int_{-1}^{1} e^{-j2\pi ft}\,dt = -\left.\frac{e^{-j2\pi ft}}{j2\pi f}\right|_{-1}^{1} = \frac{2\sin 2\pi f}{2\pi f} = 2\ \text{sinc}\ 2f$$



(c) Compare and contrast these plots (note that you can answer this question correctly without getting the correct answers for (a) and (b).

Upon observation of the above spectrums, one can see the contrast between the two plots as one having a wider spectrum and one having a narrower spectrum. Looking at the magnitude spectrum for part (b), the spectrum is narrow because it is divided by 2. Whereas, the magnitude spectrum for part (a) is wider by a multiplier of 2. There is also a difference in amplitudes as one can see. The amplitude for part (b) is higher than the amplitude for part (a).
Another observation is that we know the sharper the pulse the wider the frequency spectrum.

Problem No. 3: For the following system,



(a) Find *X*(*s*):

$$
\begin{aligned}
X(s) &= \int_0^1 (1-t)e^{-st}\,dt \\
&= \int_0^1 e^{-st} - \int_0^1 te^{-st}\,dt \\
&= -\frac{e^{-st}}{s}\Bigg|_0^1 - \left[-\frac{te^{-st}}{s}\Bigg|_0^1 + \int_0^1 \frac{e^{-st}}{s}\,dt\right] \\
&= \left[\frac{-e^{-s}}{s} + \frac{1}{s}\right] - \left[\frac{-e^{-s}}{s} - \frac{e^{-st}}{s^2}\Bigg|_0^1\right] \\
&= \frac{-e^{-s}}{s} + \frac{1}{s} - \left[\frac{-e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right] \\
X(s) &= \frac{1}{s} - \frac{1}{s^2} + \frac{e^{-s}}{s^2}
\end{aligned}
$$

(b) Find *Y*(*s*):

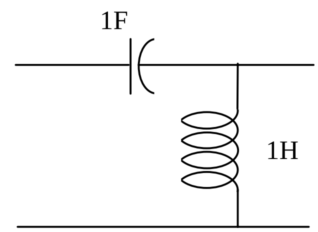
$$
\begin{aligned}
H(s) &= e^{-s}\left[X(s)\right] \\
&= e^{-s}\left[\frac{1}{s} - \frac{1}{s^2} + \frac{e^{-s}}{s^2}\right] \\
H(s) &= \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}
\end{aligned}
$$

To find Y(s) we need to use the convolution theorem.
The convolution theorem in the time domain is: $y(t) = x(t)*h(t)$
The convolution theorem in the frequency domain is: $Y(s) = X(s)H(s)$
Therefore:

$$
Y(s) = \left(\frac{1}{s} - \frac{1}{s^2} + \frac{e^{-s}}{s^2}\right)\left(\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}\right)
$$

$$
Y(s) = \frac{e^{-s}}{s^2} - \frac{2e^{-s}}{s^3} + \frac{2e^{-2s}}{s^3} + \frac{e^{-s}}{s^4} - \frac{2e^{-2s}}{s^4} + \frac{e^{-3s}}{s^4}
$$

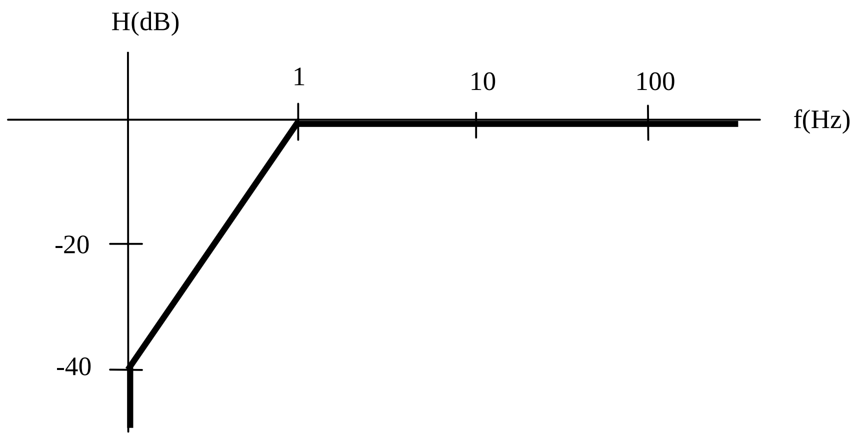
(c) For the circuit shown, construct the Bode magnitude plot.



$$X(s) = I\left(\frac{1}{s}\right) + I(s)$$
$$= I\left(\frac{1}{s} + s\right)$$
$$= I\left(\frac{1+s^2}{s}\right)$$

$$Y(s) = I(s)$$

$$T(s) = \frac{Y(s)}{X(s)} = \frac{I(s)}{I\left(\frac{1+s^2}{s}\right)}$$
$$= \frac{s^2}{s^2+1}$$



The slope of the curve is 40dB/dec.

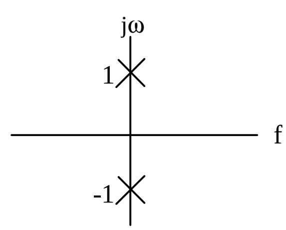
(d) How many poles and zeros does the system have?

\# of poles $\Rightarrow$ 1 complex conjugate pair at $s^2 + 1 = 0$, or at $(s - j)(s + j)$.

\# of zeros $\Rightarrow$ 2 zeros at $s^2 = 0$.

(e) Using concepts introduced in Chaps. 4 - 6, determine the stability of the system.

For a system to be stable all of the poles must lie in the LHP; if any poles are in the RHP, the system is unstable. To determine the stability of a system, one can use a Routh Array, Nyquist Criterion, or by graphing the poles as is done here.



As can be seen here, the poles lie on the $j\omega$ plane making this system **marginally stable**.