EE 3133 EXAM NO. 2

Name:

Problem	Points	Score
1a	10	
1b	10	
1c	10	
1d	10	
2a	10	
2b	10	
2c	10	
3a	10	
3b	10	
3c	10	
Total	100	

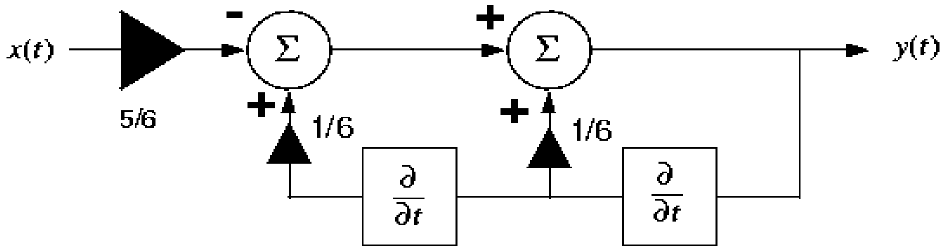
Notes:

1. The exam is closed books/closed notes - except for one page of notes.

2. Please show ALL work. Incorrect answers with no supporting explanations or work will be given no partial credit.

3. Please indicate clearly your answer to the problem. If I can't read it (and I am the judge of legibility), it is wrong. If I can't follow your solution (and I get lost easily), it is wrong. All things being equal, neat and legible work will get the higher grade:)

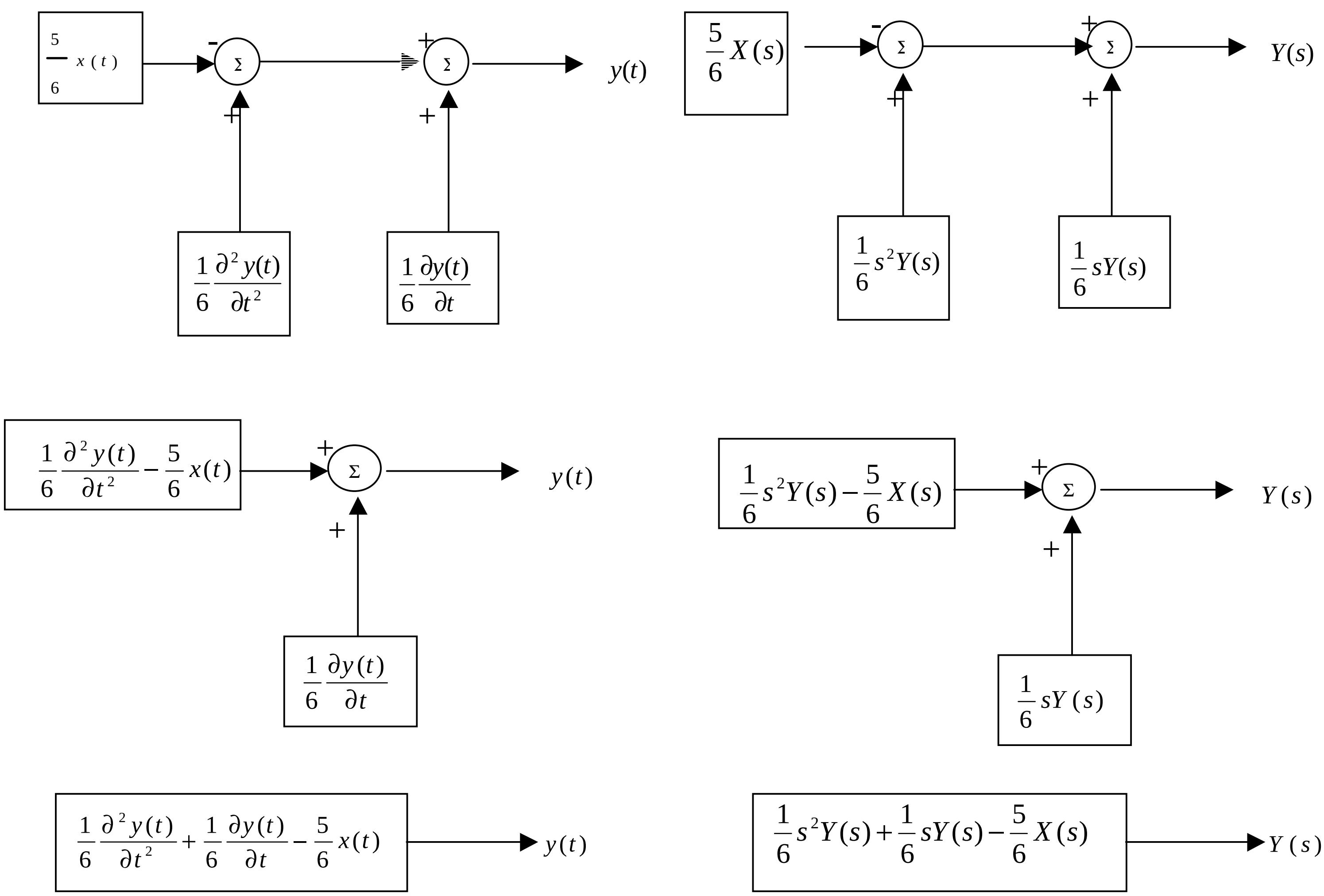
Problem No. 1: Block Diagrams



(a) Find the transfer function of this system using Laplace transforms.

The transfer function is defined by the following: $H(s) = \frac{Y(s)}{X(s)}$

The system can be solved using the time domain solution and transforming the result to obtain the transfer function in the frequency domain; or,the system can be represented in the frequency and solved directly. The Block Diagram representation is reduced to give the output Y(s) in terms of the input X(s)



The Laplace Transform of the time domain solution is

$$\therefore \frac{1}{6}\frac{\partial^2 y(t)}{\partial t^2} + \frac{1}{6}\frac{\partial y(t)}{\partial t} - \frac{5}{6}x(t) = y(t)$$

Transfer Function

$$\mathcal{L}\left\{\frac{1}{6}\frac{\partial^2 y(t)}{\partial t^2} + \frac{1}{6}\frac{\partial y(t)}{\partial t} - \frac{5}{6}x(t)\right\} = \mathcal{L}\{y(t)\}$$

$$\frac{Y(s)}{X(s)} = \frac{5}{6}\,\frac{1}{\left(\frac{1}{6}s^2 + \frac{1}{6}s - 1\right)} = \frac{5}{\left(s^2 + s - 6\right)}$$

$$\frac{1}{6}s^2Y(s) + \frac{1}{6}sY(s) - Y(s) = \frac{5}{6}X(s)$$

(b) Find the impulse response

The impulse response of a system is the output due to the application of the unit impulse to the input of the system. In the time domain, the impulse response is given by y(t) = x(t)*h(t).

With the Laplace Transform Theorem from Table 5-2 (Convolution), the solution in the frequency domain is represented by Y(s) = X(s)H(s).

$$Y(s) = H(s)X(s) = \frac{5}{\left(s^2 + s - 6\right)}X(s) = \frac{5}{(s+3)(s-2)} = \frac{A}{(s+3)} + \frac{B}{(S-2)}$$

Note:
$x(t) = \delta(t) \Leftrightarrow X(s) = 1$

The coefficients A and B may be found with the use of the Heaviside's Expansion Theorem.

$$A = (s+3)Y(s)\Big|_{s=-3} = \frac{5}{(-3-2)} = -1 \qquad B = (s-2)Y(s)\Big|_{s=2} = \frac{5}{(2+3)} = 1$$

Which gives $Y(s) = \frac{-1}{(s+3)} + \frac{1}{(S-2)}$

Now, the time domain solution is obtained with the use of the inverse Laplace Transform .

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-1}{(s+3)} + \frac{1}{(S-2)}\right\} \quad \Rightarrow \quad y(t) = \left(e^{2t} - e^{-3t}\right)u(t)$$

(c) Determine whether the system is stable or unstable. Show ALL work --- the correct answer with no supporting work gets no points. Be as detailed as possible.

The stability of the system may be found with the use of the Routh Array. The Routh array is constructed with the coefficients of the characteristic polynomial that has the form $s^n + s^{n-1} + s^{n-2} + \ldots + s^0$. The number of sign changes in the first column represents the number of Right Hand Poles (RHP) of the system.

$$\begin{array}{l|lll}
s^n & a_n & a_{n-2} & a_{n-4} \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} \\
s^{n-2} & b_1 & b_2 & \\
s^{n-3} & c_1 & c_2 &
\end{array}$$

The a_n (a_n, a_{n-1}, a_{n-2} ....) entries are the coefficents of the characteristic polynomial
$s^n + s^{n-1} + s^{n-2} + \ldots + s^0$

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \qquad b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$c_1 = \frac{b_1 a_{n-3} - b_2 a_{n-1}}{b_1} \qquad c_2 = \frac{b_1 a_{n-5} - b_3 a_{n-1}}{b_1}$$

For the system

$$H(s) = \frac{5}{\left(s^2 + s - 6\right)}$$

$$\begin{array}{l|ll}
s^2 & 1 & -6 \\
s^1 & 1 & \\
s^0 & -6 &
\end{array}$$

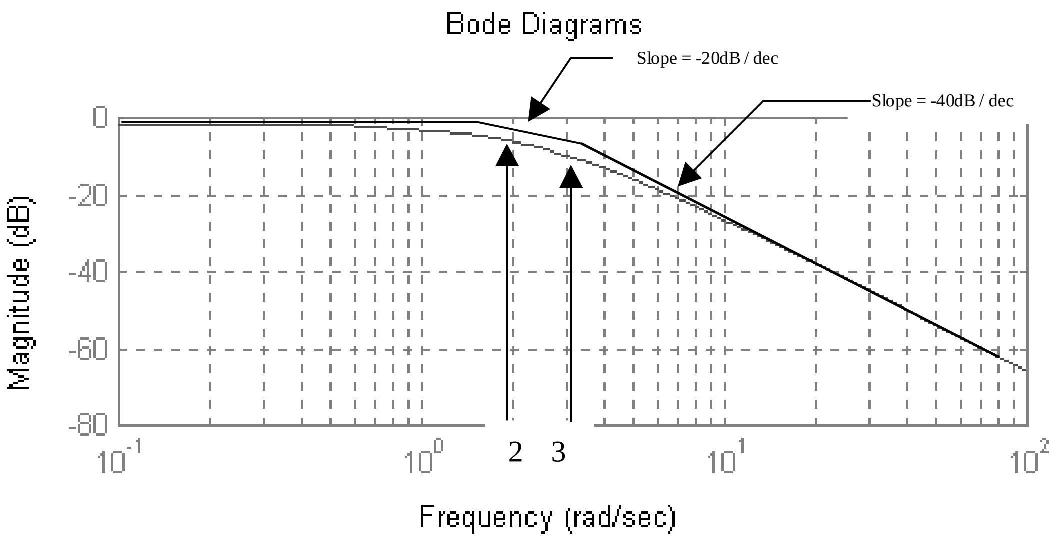
$$b_1 = \frac{-6-0}{1} = -6$$

Sign Change
The system contains one RHP

Therefor the system is unstable.

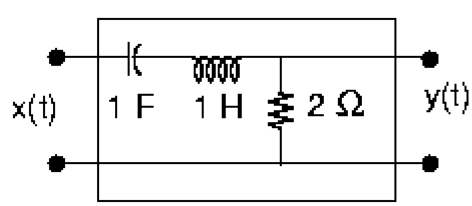
(d) Sketch the frequency response (magnitude only) of the system using Bode plots.

The transfer function of the system is $H(s) = \dfrac{5}{(s+3)(s-2)}$



Problem No. 2: Circuit analysis using Fourier transforms.

For the circuit shown below:



(a) State all the Fourier transform theorems that are invoked when you compute transfer function of this circuit. You must give specific evidence to support each theorem described (and I must be able to understand your logic!).

The systems input output relationship can be described by the use KVL.

$$x(t) = \frac{1}{C}\int_{-\infty}^{t} i(\lambda)\,d\lambda + L\frac{\partial i(t)}{dt} + i(t) * R \qquad i(t) = \frac{y(t)}{R}$$

With the Linearity theorem $a_1x_1(t) + a_2x_2(t) \Leftrightarrow a_1X_1(f) + a_2X_2(f)$

$$\mathcal{L}\{x(t)\} = \mathcal{L}\left\{\frac{1}{C}\int_{-\infty}^{t} i(\lambda)d\lambda + L\frac{\partial i(t)}{dt} + i(t) * R\right\} \Rightarrow \mathcal{L}\{x(t)\} = \mathcal{L}\left\{\frac{1}{C}\int_{-\infty}^{t} i(\lambda)d\lambda\right\} + \mathcal{L}\left\{L\frac{\partial i(t)}{dt}\right\} + \mathcal{L}\{i(t) * R\}$$

$$\mathcal{L}\{i(t)\} = \mathcal{L}\left\{\frac{y(t)}{R}\right\}$$

Now, we make use of the differentiation and integration Theorems to obtain the transfer function.

$$\mathcal{L}\left\{\frac{1}{C}\int_{-\infty}^{t} i(\lambda)d\lambda\right\} = \frac{Y(s)}{C(j2\pi f)} \qquad \mathcal{L}\left\{L\frac{\partial i(t)}{dt}\right\} = L(j2\pi f)^1 Y(s)$$

$$X(s) = Y(s)\left(\frac{1}{RC\,(j2\pi f)} + \frac{L}{R}(j2\pi f) + 1\right)$$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{1}{\left(\frac{1}{RC(j2\pi f)} + \frac{L}{R}(j2\pi f) + 1\right)} = \frac{RC(j2\pi f)}{1 + LC(j2\pi f)^2 + RC(j2\pi f)} = \frac{RC(j2\pi f)}{1 + 2(j2\pi f) + (j2\pi f)^2} = \frac{2(j2\pi f)}{(1 + (j2\pi f))^2}$$

(c) State and prove the Frequency Translation Theorem.

The frequency translation theorem follows by multiplying x(t) with $e^{j(2\pi f_o)t}$ and then applying the transform of the product x(t) $e^{j(2\pi f_o)t}$

$$\int_{-\infty}^{\infty} x(t)\; e^{j(2\pi f_o)t}\, e^{-j(2\pi f)t}\, dt = \int_{-\infty}^{\infty} x(t)\; e^{-j2\pi(f - f_o)t}\, dt = X(f - f_o)$$

As we can see, the result of multiplying the time domain function x(t) with $e^{j(2\pi f_o)t}$ will produce a shift in the frequency domain.

(d) Find the impulse response of the circuit using Fourier Transforms.

$$H(f) = \frac{Y(f)}{X(f)} = \frac{1}{\left(\frac{1}{RC(j2\pi f)} + \frac{L}{R}(j2\pi f) + 1\right)} = \frac{RC(j2\pi f)}{1 + LC(j2\pi f)^2 + RC(j2\pi f)} = \frac{RC(j2\pi f)}{1 + 2(j2\pi f) + (j2\pi f)^2} = \frac{2(j2\pi f)}{(1 + (j2\pi f))^2}$$

The transfer function has the form

$$H(f) = \frac{Y(f)}{X(f)} = \frac{2(j2\pi f)}{(1 + (j2\pi f))^2} = 2(j2\pi f)G(f) \qquad \frac{1}{(1 + (j2\pi f))^2} = G(f)$$

From Table 4-1 Fourier Transform Theorems (Differentiation Transform Pair)

$$\frac{\partial^n g(t)}{\partial^n d} \Leftrightarrow (j2\pi f)^n G(f)$$

And from Table 4-2 Fourier Transform Pairs (pair 5)

$$te^{(-\alpha t)}u(t) \Leftrightarrow \frac{1}{(1 + (j2\pi f))^2}$$

Therefor we get

$$\frac{\partial te^{(-\alpha t)}}{\partial t} = -\alpha te^{-\alpha t} + e^{-\alpha t}$$

Problem No. 3: The Dreaded Thought Problem

Signal to Noise (SNR) ratio is defined as the ratio of the power of a signal and the powe of the noise in a system, computed on a log scale and measured in dB:

$$SNR|_{dB} = 10\log_{10}\left(\frac{P_{signal}}{P_{noise}}\right)$$

Assume the signal is given by $x(t) = \sin\omega_o t$,and the noise is given by $w(t-nT_o) = e^{-\alpha|t-nT_o|}$.

(a) Compute SNR in the time domain.

$$P_{x(t)} = \frac{1}{T_o}\int_{t_o}^{t_o+T_o} |\sin \omega_o t|^2\, dt = \frac{1}{2T_o}\int (1 - \cos 2\omega_o t)\, dt = \frac{1}{2T_o}\left[t - \frac{\sin 2\omega_o t}{2\omega_o}\right] = \frac{1}{2}$$

$$P_{w(t)} = \frac{1}{T_o}\int_{t_o}^{t_o+T_o} \left|e^{-\alpha t}\right|^2 dt = \frac{1}{1}\int_0^1 e^{-2\alpha t} dt = \frac{1}{-2\alpha}\left[e^{-2\alpha t}\right]_0 = \frac{1-e^{-2\alpha}}{2\alpha}$$

$$SNR_{dB} = 10\log_{10}\left(\frac{P_{x(t)}}{P_{w(t)}}\right) = 10\log_{10}\left(\frac{\alpha}{1-e^{-2\alpha}}\right)$$

(b) Compute SNR in the frequency domain and prove it is equivalent to the time domain calculation.

Parseval's Theorem states that the power of a time domain signal is equal to the power in the frequency domain as given by.

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2\, dt = \int_{-\infty}^{\infty}|X(f)|^2\, df$$

$$P_{X(f)} = \int_0^1\left|\frac{1}{2j}\delta(f-f_o) - \frac{1}{2j}\delta(f+f_o)\right|^2 df = \int\frac{1}{(2j)^2}\delta(f-f_o)^2\, df - \int\frac{1}{(2j)^2}\delta(f+f_o)^2\, df$$

Now, with equation (1-56) p. 27 SIGNALS AND SYSTEMS

$$\int_{t_1}^{t_2} x(t)\delta(t-t_o)dt = \begin{cases} x(t_o), & \mathrm{t}_1 < t_o < t_2 \\ 0, & \end{cases}$$

The integral evaluates to

$$P_{X(f)} = \int \frac{1}{4}\delta(f - f_o)^2 df + \int \frac{1}{4}\delta(f + f_o)^2 df = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}W$$

Now, for the noise signal

Parsevals Theorem states that the power of a time domain signal is equal to the power in the frequency domain as given by.

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2 dt = \int_{-\infty}^{\infty}|W(f)|^2 df$$

With $\Im\{x(t)\} = W(f)$

In this case, we have for one period of the original signal.

$$\Im\{x(t)\} = W(f) = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$$ Table 4-2 Pair number 6.

The power is therefore given by

$$P_{W(f)} = \int_{-\infty}^{\infty}|W(f)|^2 df = 2\int_0^{\infty}\left|\frac{2\alpha}{\alpha^2 + (2\pi f)^2}\right|^2 df = 2\int_0^{\infty}\frac{4\alpha^2}{\left(\alpha^2 + (2\pi f)^2\right)^2} df$$

The integral can be calculated with the use of substitution and the help of an integral Table.

With $\omega = 2\pi f$

$d\omega = 2\pi df$

$df = d\omega/2\pi$

$$P_{W(f)} = \frac{1}{\pi}\int_0^\infty \frac{4\alpha^2}{\left(\alpha^2+(\omega)^2\right)^2}\,d\omega$$

From a table of integrals,

$$P_{W(f)} = \frac{4\alpha^2}{\pi}\int_0^\infty \frac{d\omega}{\left(\alpha^2+(\omega)^2\right)^2} = \frac{4\alpha^2}{\pi}\left[\frac{\omega}{2\alpha^2\left(\alpha^2+\omega^2\right)} + \frac{1}{\alpha^3}\tan^{-1}\frac{\omega}{\alpha}\right]_0^\infty = \frac{4\alpha^2}{\pi}\left[\left(0+\frac{\pi}{4\alpha^3}\right)-\left(0-\frac{\pi}{4\alpha^3}\right)\right]$$

$$= \frac{4\alpha^2}{\pi}\left[\left(\frac{\pi}{4\alpha^3}\right)+\left(\frac{\pi}{4\alpha^3}\right)\right] = \frac{4\alpha^2}{\pi}\left[\frac{2\pi}{4\alpha^3}\right] = \frac{2}{\alpha}$$

Notice, we used the fact that

$$\lim_{f\to\infty}\tan^{-1}\left(\frac{\omega f}{\alpha}\right) = \frac{\pi}{2}$$

Therefore the SNR is

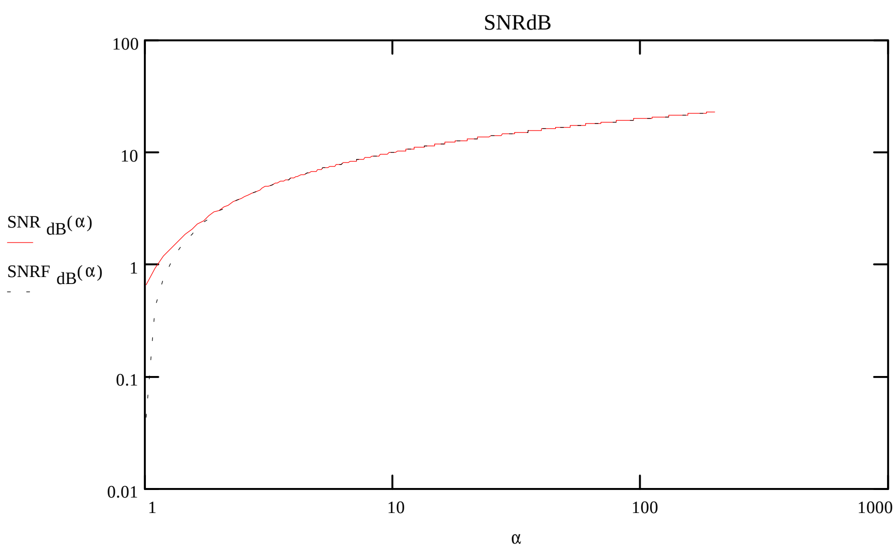
$$SNR = 10\log_{10}\left(\frac{P_{X(f)}}{P_{W(f)}}\right) = 10\log_{10}\left(\frac{1/2}{2/\alpha}\right) = 10\log_{10}(\alpha)$$

(c) Explain how the SNR varies with ω and α .

The SNR is independent of the variable ω as given by the time and frequency domain solutions above. However, the SNR is a function of the variable α. The following plot shows the relationship of the SNR of both the time domain and the frequency domain solutions as a function of ω. The plot also shows the equality of the two solutions.

$\alpha := 1.01, 1.1 .. 200$

$SNR_{dB}(\alpha) := 10 \cdot \log\left(\frac{\alpha}{1 - e^{-2\cdot\alpha}}\right)$ $SNRF_{dB}(\alpha) := 10 \cdot \log(\alpha)$



The SNR is directly proportional to α and increases linearly as α increases.