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Problem	Points	Score
1a	10	
1b	10	
1c	10	
2a	10	
2b	10	
2c	10	
2d	10	
3a	10	
3b	10	
3c	10	
Total	100	

Notes:

1. The exam is closed books/closed notes - except for one page of notes.

2. Please show ALL work. Incorrect answers with no supporting explanations or work will be given no partial credit.

3. Please indicate clearly your answer to the problem.

Problem No. 1: For the linear time-invariant system: $H(s) = \dfrac{s+2}{s^2 - 2s - 3}$

(a) Find the state variable description of the system.

The transfer function of the system has the form of the ratio of two polynomials as given by

$$\frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \ldots + b_1 s^1 + b_o}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_1 s^1 + a_o} \qquad m < n$$

Y(s) and U(s) is the Laplace transform of the time domain system output and input respectively. The denominator can be factored to give the relationship of H(s) as follows

$$\frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \ldots + b_1 s^1 + b_o}{(s - p_1)(s - p_2)\ldots(s - p_n)}$$

Now, we can use partial fraction expansion and obtain

$$\frac{Y(s)}{U(s)} = \frac{B_1}{s - p_1} + \frac{B_2}{s - p_2} + \ldots \frac{B_n}{s - p_n} \qquad X_i(s) = \frac{U(s)}{s - p_i}$$

So, with the use of the inverse laplace transform, we can determine the state variables as

$$\dot{x}_i = x_i + u$$

Which we can write in matrix form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} p_1 & 0 & 0 & \cdots & 0 \\ 0 & p_2 & 0 & \cdots & 0 \\ \vdots & & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} B_1 & B_1 & \cdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

For the system : $H(s) = \dfrac{s+2}{s^2 - 2s - 3}$

$$A = (s-3)\frac{Y(s)}{U(s)}\bigg|_{s=3} = \frac{5}{4}$$

$$\frac{Y(s)}{U(s)} = \frac{s+2}{s^2-2s-3} = \frac{A}{s-3} + \frac{B}{s+1}$$

$$B = (s+1)\frac{Y(s)}{U(s)}\bigg|_{s=-1} = \frac{-1}{4}$$

$$\therefore \frac{Y(s)}{U(s)} == \frac{5/4}{s-3} + \frac{-1/4}{s+1}$$

Therefor, we identify the state variable matrix as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \qquad \mathbf{y} = \begin{bmatrix} \frac{5}{4} & \frac{-1}{4} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b) Compute the state transition matrix, $\mathbf{\Phi}(t)$.

We can find the state transition matrix with $\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1}$ After we form the matrix, we can take the inverse Laplace transform to obtain $\mathbf{\Phi}(t) = \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\} = e^{\mathbf{A}t}$

With

$$(s\mathbf{I} - \mathbf{A}) = \left\{ s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \right\} = \begin{bmatrix} s-3 & 0 \\ 0 & s+1 \end{bmatrix}$$

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{adj(\mathbf{A})}{|\mathbf{A}|} = \frac{1}{(s+1)(s-3)}\begin{bmatrix} s+1 & 0 \\ 0 & s-3 \end{bmatrix} = \begin{bmatrix} \frac{1}{s-3} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

$$\Phi(t) = e^{\mathbf{A}t} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix}$$

(c) Using the state variable representation, implement this as an RLC circuit.

Due to fact that the system is unstable, the system can not be implemented as a RLC circuit.

Problem N0. 2: This problem deals with various aspects of Z-Transforms.

(a) **Derive** the expression for the Z-Transform of $x(n) = na^{-n}\,u(n)$.

With the definition of the Z–Transform given as:

$$X(z) = \sum_{n=0}^{\infty} x(nT) z^{-n}$$

We have the Z–Transform of x(n)

$$X(z) = \sum_{n=0}^{\infty} na^{-n}$$

Now, we differentiate with respect to z to obtain:

$$\frac{d}{dz}X(z) = \sum_{n=0}^{\infty} x(nT)(-n)z^{-n-1}$$

Therefor, we can multiply both sides by $-z^{-1}$ so we can obtain a form that can be recognized in Table 8-1 .

$$-z\frac{d}{dz}X(z) = \sum_{n=0}^{\infty}[nx(nT)]z^{-n}$$

Now, we can use Table 8-1 entry 3 to make the transform:

$$\sum_{n=0}^{\infty} e^{-anT}z^{-n} = \frac{1}{1-e^{-aT}z^{-1}}$$

With the Quotient Rule, we differentiate with respect to z and obtain:

$$\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \longrightarrow \sum_{n=0}^{\infty} e^{-anT}(-n)z^{-n-1} = \frac{-e^{-aT}z^{-2}}{\left(1-e^{-aT}z^{-1}\right)^2}$$

Finally, we multiply by –z to give the original identity :

$$\sum_{n=0}^{\infty} ne^{-anT}z^{-n} = \frac{e^{aT}z^{-1}}{\left(1-e^{-aT}z^{-1}\right)^2}$$

We note that this gives us a general result of the solution and we must therefor manipulate the expression to give the original identity. We can multiply by T and let a approach 0 which gives:

$$\sum_{n=0}^{\infty}(nT)z^{-n} = \frac{Tz^{-1}}{(1-z^{-1})^2}$$

(b) For the transfer function, $H(z) = \dfrac{1-(1/2)z^{-1}}{1-(3/2)z^{-1}-z^{-2}}$, find a closed form expression for h(n) (dont't use long division).

$$H(z) := \frac{1-\frac{1}{2}\cdot z^{-1}}{1-\left(\frac{3}{2}\cdot z^{-1}\right)-z^{-2}}$$

simplifies to

$$H(z) := (2\cdot z-1)\cdot\frac{z}{\left(2\cdot z^2-3\cdot z-2\right)}$$

Now, we can make the substitution

$$X(z) := \frac{H(z)}{z}$$

$$X(z) := \frac{(2\cdot z-1)}{\left(2\cdot z^2-3\cdot z-2\right)}$$

by factoring, yields

$$X(z) := \frac{(2\cdot z-1)}{((2\cdot z+1)\cdot(z-2))}$$

Now, we can factor out 2 from the first term ii
the denominator and obtain

$$X(z) := \frac{(2\cdot z-1)}{2\cdot\left(z+\frac{1}{2}\right)\cdot(z-2)}$$

yields

$$X(z) := \frac{\left(z - \frac{1}{2}\right)}{\left(z + \frac{1}{2}\right)\cdot(z-2)} \longrightarrow$$

expands in partial fractions to

$$X(z) := \frac{K_1}{z + \frac{1}{2}} + \frac{K_2}{z-2}$$

$$K_1(z) := X(z)\cdot\left(z + \frac{1}{2}\right)$$

yields

$$K_1(z) := \frac{\left(z - \frac{1}{2}\right)}{\left(z + \frac{1}{2}\right)\cdot(z-2)}\cdot\left(z + \frac{1}{2}\right)$$

simplifies to

$$K_1(z) := \frac{1}{2}\cdot\frac{(2\cdot z - 1)}{(z-2)}$$

$$K_1\left(-\frac{1}{2}\right) = 0.4$$

$$K_2(z) := X(z)\cdot(z-2)$$

yields

$$K_2(z) := \frac{\left(z - \frac{1}{2}\right)}{\left(z + \frac{1}{2}\right)\cdot(z-2)}\cdot(z-2)$$

simplifies to

$$K_2(z) := \frac{(2\cdot z - 1)}{(2\cdot z + 1)}$$

$$K_2(2) = 0.6$$

Therefor, H(z) is of the form

$$H(z) := \frac{0.4\cdot z}{z + \frac{1}{2}} + \frac{0.6\cdot z}{z-2} \longrightarrow H(z) := \frac{0.4}{1 + \frac{1}{2}\cdot z^{-1}} + \frac{0.6}{1 - 2\cdot z^{-1}}$$

Now, we can inverse transform using transform pairs 1 and 3 to obtain h(n).

$$h(nT) = -0.4(0.5)^n + 0.6(2)^n \qquad \text{for } n \geq 0$$

(c) Is the system stable?

No: The solutions for the two poles show that the system is unstable.

$$1 + \frac{1}{2} \cdot z^{-1}$$

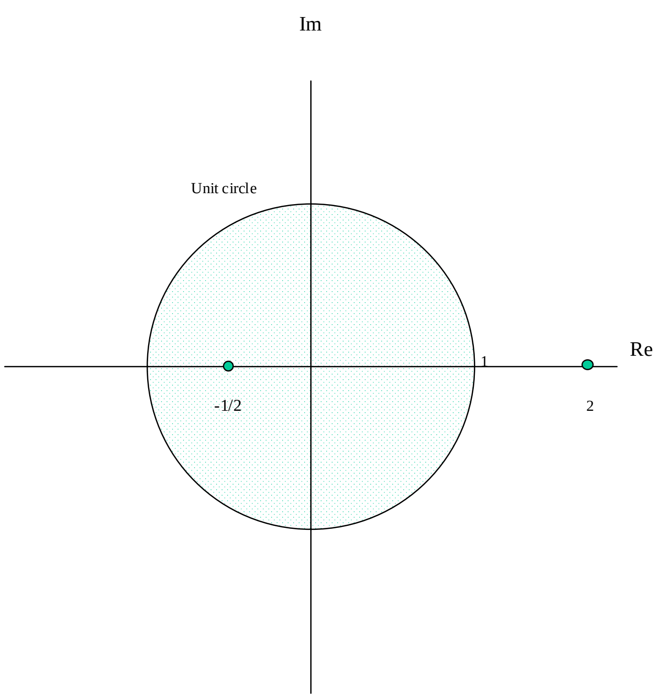
has solution

$$\frac{-1}{2}$$

$$1 - 2 \cdot z^{-1}$$

has solution

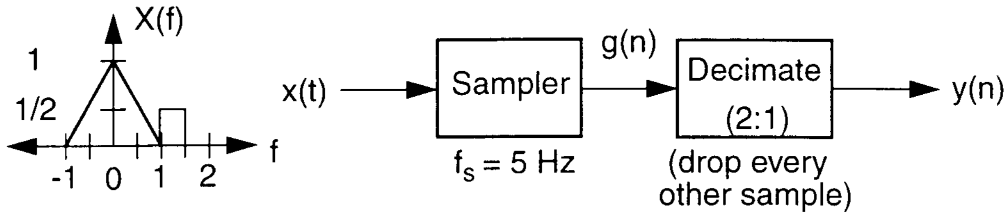
$$2$$



(d) Is the system causal? Explain.

Yes. From $h(nT) = -0.4(0.5)^n + 0.6(2)^n$ for $n \geq 0$ we see that the output of the system will be 0 for any input x(nT) for n < 0.

Problem No. 3: For the system shown:



(a) Plot the spectrum of g(n)

The expression for the spectrum of the sampled signal x(t) is given by:

$$X_s = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

The spectrum of the sampled signal is obtained by multiplying X(f) by f_s and reproducing the Product f_s X(f) about dc and all harmonics of the sampling frequency f_s.

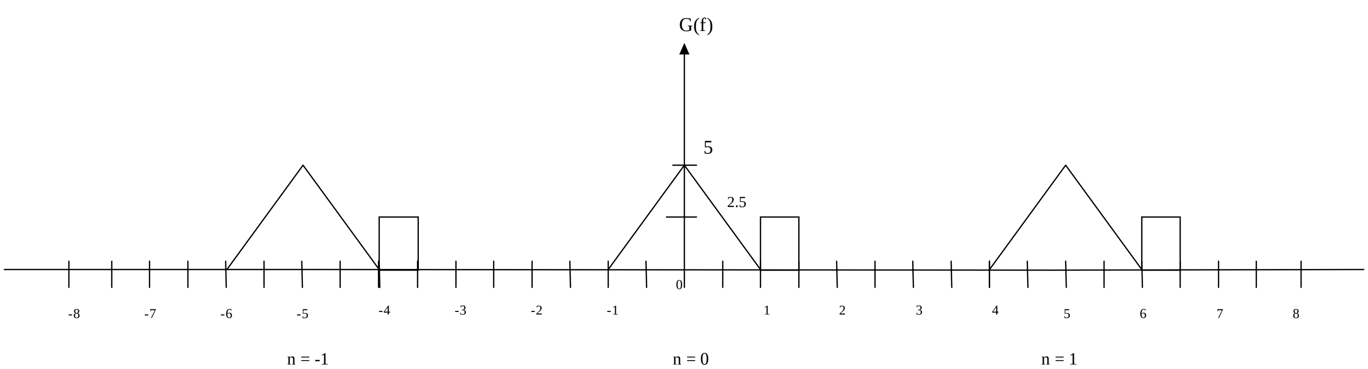
For n = 0

$$X_s = 5X(f - 0)$$

For n = 1

$$X_s = 5X(f - 5)$$

So, we have



(b) Plot the spectrum of y(n):

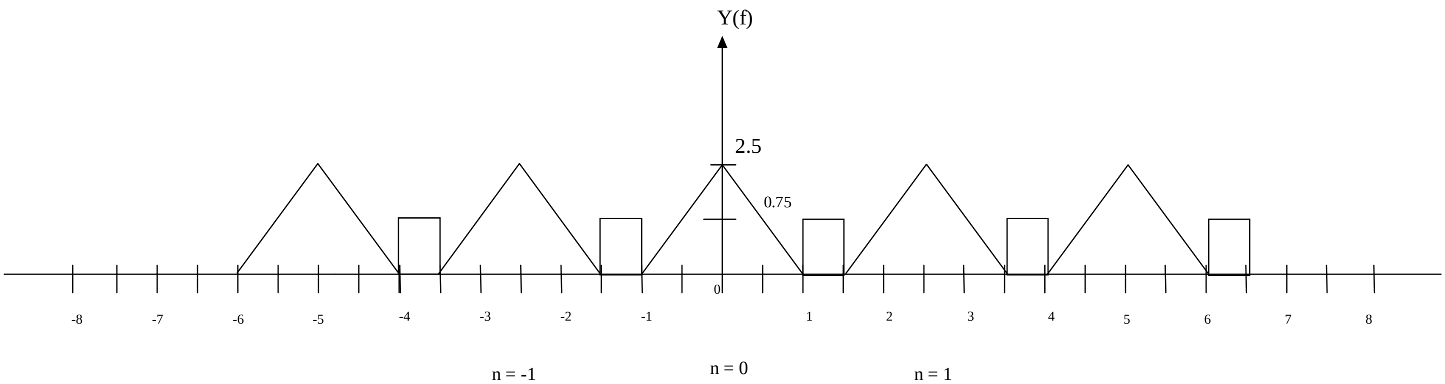
The signal y(n) is the sample of x(t) at the sample rate of 5 Hz/2 = 2.5 Hz.

For n = 0

$$X_s = 2.5X(f-0)$$

For n = 1

$$X_s = 2.5X(f-2.5)$$



(b) How do you explain the fact that the sample frequency of y(n) is less than the Nyquist rate, yet there is no distortion?

From the original figure, we can see that the highest frequency is 1.5 Hz so the Nyquist rate is 2*1.5 = 3 Hz. However, the original signal width is 2.5Hz, so, the signal can be reconstructed with no distortion due the fact that the sampled signal is reproduced about dc and all harmonics of the sampling frequency f_s.