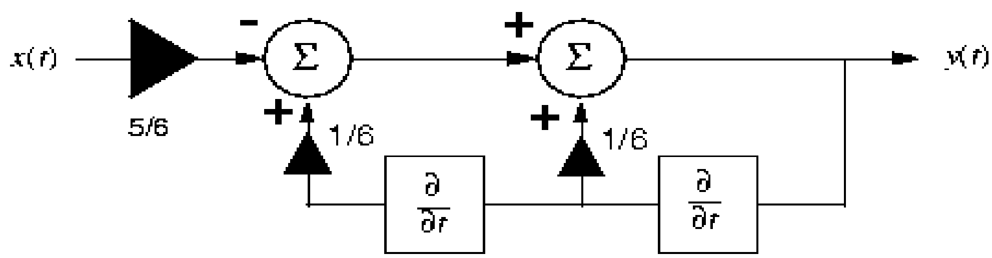
Problem No. 1: Block Diagrams



(a) Find the transfer function of this system using Laplace transforms.

Let the output of the first summer = E1(t).

E1(t) = -5/6x(t) + 1/6 $d^2y(t)/dt$

y(t) = E1(t) + 1/6 dy(t)/dt

Substituting for E1(t) => y(t) = 1/6 $d^2y(t)/dt$ + 1/6 dy(t)/dt - 5/6x(t)

Now convert to Laplace domain:

Y(s) = 1/6 s2Y(s) + 1/6sY(s) - 5/6X(s)

Multiplying by 1/Y(s) gives... S2/6 + s/6 - 5 X(s)/Y(s) = 1

=> 5X(s)/Y(s) = s2/6 + s/5 - 1 => 5X(s) = [s2/6 + s/6 - 1]Y(s)

e need the form Y(s)/X(s) so the transfer function

H(s) = Y(s)/X(s) = 30 / [s^2 + s - 6]

(b) Find the impulse response.

For the impulse response, x(t) = δ(t). Therefore X(s) = 1. The output response Y(s) = H(s)X(s) which is simply H(s). So now convert Y(s) to the time domain using inverse Laplace transforms.

First we simplify the denominator by using partial fractions.

30 / [s^2 + s - 6] = 30 / [(s+3) (s-2)] = a/(s+3) + b/(s-2)

$a = 30/(s-2)\,|_{s=-3}$ $a = -5$ $b = 30/(s+3)\,|_{s=2}$ $b = 6$

this gives $Y(s) = -5/(s+3) + 6/(s-2)$

Which transforming to the time domain yields.....

$$\mathbf{y(t) = [-5e^{-3t} + 6e^{2t}]u(t)}$$

(c) Determine whether the system is stable or unstable.

For a system to be stable, the poles of the transfer function need to lie in the left half portion of the s-plane. For transfer functions containing polynomials, the roots are determined by factoring the polynomial in the denominator (provided the degree of the numerator polynomial is equal to or less than the degree in the denominator). For large degree polynomials, the number of roots in the r.h.p. can be determined using the **Routh Array**. For this case, the denominator can be easily factored and the roots determined.

The denominator is $s^2 + s - 6$ Which factors to $(s + 3)(s - 2)$

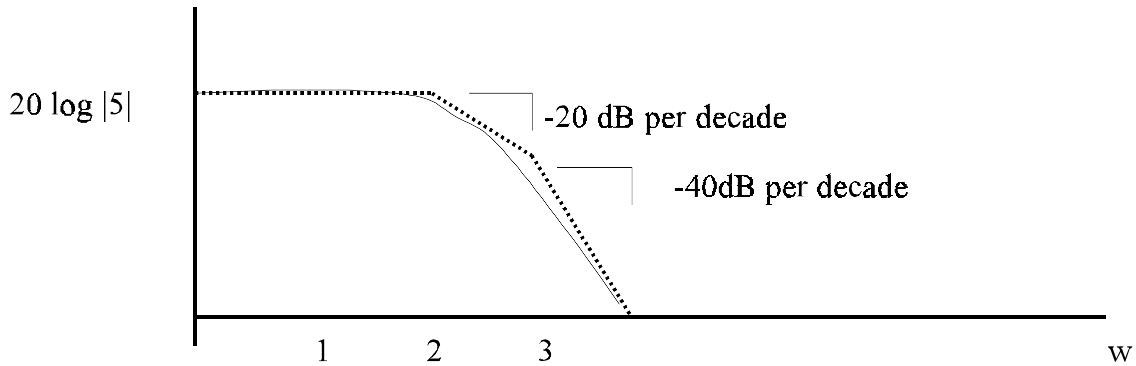
There is a root in the r.h.p. at $s = 2$ therefore the system is **unstable.**

(d) Sketch the frequency response (magnitude only) of the system using Bode plots.

Putting the equation $30 / [(s+3)(s-2)]$ in standard form gives:

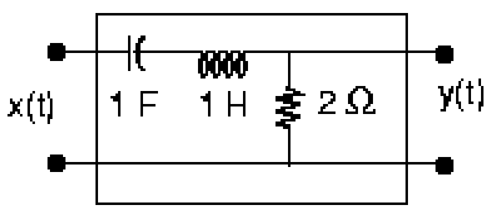
$$5 / [(s/3 + 1)(s/2 - 1)] \qquad s = jw$$

this results in a pole at $w = 3$ and $w = 2$ with a constant term of $20\log|5|$.



Problem No. 2: Circuit analysis using Fourier transforms.

For the circuit shown below:



(a) State all the Fourier transform theorems that are invoked when you compute the transfer function of this circuit. You must give specific evidence to support each theorem described (and I must be able to understand your logic!).

First note $i(t) = y(t)/2$ and $L = 1$, $C = 1$.

Using KVL: $1/c\int y(t)/2\ dt + L/2dy(t)/dt + y(t) = x(t)$

Inserting C and L => $\int y(t)/2\ dt + 1/2dy(t)/dt + y(t) = x(t)$

Using the Integration Theorem, Differentiation Theorem, and the Fourier transfor pair $X(f) = x(t)$ yields...

$Y(f) / 2(j2\pi f) + ½ (j2\pi f)Y(f) + Y(f) = X(f)$ Factoring out $Y(f)$ gives

$Y(f)[1 / 2(j2\pi f) + (j2\pi f)/2 + 1] = X(f)$ The transfer function $H(f) = Y(f)/X(f)$ Therefore

$H(f) = Y(f)/X(f) = 1 / [1 / 2(j2\pi f) + (j2\pi f)/2 + 1]$ Multiply by $2j2\pi f / 2j2\pi f$ gives the standard form.

$$\mathbf{H(f) = 2(j2\pi f) / [(j2\pi f)^2 + 2(j2\pi f) + 1]}$$

(b) State and prove the frequency translation theorem.

The Frequency Translation Theorem states

$\mathscr{F}\{ x(t)e^{j2pifot} \} = X(f - fo)$ Which simply means a shift in frequency in the Fourier domain

is equal to a time-delay in the time domain. The proof is as follows...

Using the Fourier integral, the expression for the Fourier transform of $x(t)e^{j2pit}$ is written as

$\int x(t)e^{j2pifot}\ e^{-j2pift}\ dt = \int x(t)e^{-j2pi(f - fo)t}\ dt$

Because the Fourier is of the form $X(f) = \int x(t)\ e^{-j2pift}\ dt$ Substituting f for (f-fo) proves

$\mathscr{F}\{ x(t)e^{j2pifot} \} = X(f - fo)$

(c) Find the impulse response of the circuit using Fourier Transforms.

For simplification, Fourier analysis will be used to find the current i(t).

Using KVL: $1/c\int i(t)\ dt + Ld\ i(t)/dt + Ri(t) = x(t)$

Differentiating

$$i(t) + d^2i(t)/dt + Rdi(t)/dt = x(t)$$

Using the Differentiation Theorem gives

$$I(f) + (j2\pi f)^2\, I(f) + R(j2\pi f)I(f) = X(f)$$

For the impulse response $X(t) = \delta(t) \Rightarrow X(f) = 1$

Factoring out I(f) gives...

$$I(f) = 1 / [\ (j2\pi f)^2 + 2((j2\pi f) + 1\ ]$$

$$I(f) = 1 / (j2\pi f + 1)^2$$

Which is of the form $1 / (\ \alpha + j2\pi f)^2$

Letting $\alpha = 1$ transforming to the time domain gives

$$i(t) = te^{-t}u(t)$$

$$y(t) = 2\ I(t)$$

$$\mathbf{y(t) = 2\ te^{-t}u(t)}$$

Problem # 3

A signal is given by x(t) = sinwot and noise is given by w(t) = exp(-α|t-n|) n<= t <= n+1

for n = ..., -2, -1, 0, 1, 2

Compute the SNR.

The power of a sine wave is Psignal = $A^2 / 2$

The power of the noise is calculated as follows....

$$w(t) = \exp(-\alpha|t-n|)$$

$$\text{Pnoise} = 1/T \int |e^{-\alpha t}|^2 \, dt \quad \text{from 0 to 1}$$

$$\text{Pnoise} = 1/1 \int e^{-2\alpha|t|} \, dt$$

$$\text{Pnoise} = -e^{-2\alpha}/2\alpha + 1/2\alpha$$

The ratio of signal to noise is

$$\text{Psignal/PNoise} = A^2\alpha/(e^{-2\alpha} + 1)$$

SNR is given as $10\log_{10} (A^2\alpha/(e^{-2\alpha} + 1))$

(b) compute SNR in the frequency domain and prove it is equivilent to the time domain calculation.

The power of a sine wave in the frequency domain can be derived from the power spectral

density. The power spectral density is this case show the total averages power as an impulse function at -fo and fo of magnitude $A^2/4$. i.e. $A^2/4\delta(f-fo)$ and $A^2/4\delta(f+fo)$.

The total average is the sum of the two.. 2 $A^2/4$ or $A^2/2$.

By similar means the power of the noise signal can be found shown tobe the same in the frequency domain as the time domain.

Since the powers are the same in either domain, the SNR will be the same.

(c) Explain how the SNR varies with wo and α.

From the equation showing the signal to noise ratio, wo does not affect noise. α directly affects noise. As α increases, the SNR decreases. Conceptually, as α increases, the faster the exponential function of the noise approaches zero thus causing less interference with the signal.