Problem No. 1: Modeling Concepts

(a) Prove whether the signal $x(t) = te^{-\alpha t}u(t)$ is an energy signal or power signal.

$$E = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, dt$$

Due to the u(t) in the equation for x(t), the integral can be evaluated from zero to infinity.

$$E = \int_0^\infty t^2 e^{-2\alpha t} dt = \frac{-t^2}{2\alpha} e^{-2\alpha t} - \frac{2t}{4\alpha^2} e^{-2\alpha t} - \frac{t}{8\alpha^3} e^{-2\alpha t} - \frac{1}{16\alpha^4} e^{-2\alpha t} \Big|_0^\infty$$

For the upper limit, all terms evaluate to zero due to the negative exponential term (assuming $\alpha > 0$). For the lower limit, all terms involving a power of t equal zero. This leaves only one term:

$$E = \frac{1}{16\alpha^4}$$

So, energy is finite. This must be an energy signal. This implies that the power is equal to zero. As a check, we can calculate the power of the signal:

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_0^T t^2 e^{-2\alpha t} dt = \lim_{T \to \infty} \frac{1}{2T} \left[\frac{-t^2}{2\alpha} e^{-2\alpha t} - \frac{2t}{4\alpha^2} e^{-2\alpha t} - \frac{t}{8\alpha^3} e^{-2\alpha t} - \frac{1}{16\alpha^4} e^{-2\alpha t} \right]\Big|_0^T$$

Using the same arguments as above, there is only one term left after integration:

$$P = \lim_{T \to \infty} \frac{1}{2T} \left[\frac{1}{16\alpha^4} \right] = 0$$

(b) Is the signal $x(t) = \sin^2 \omega_0 t$ periodic? If so, what is its period? If not, explain.

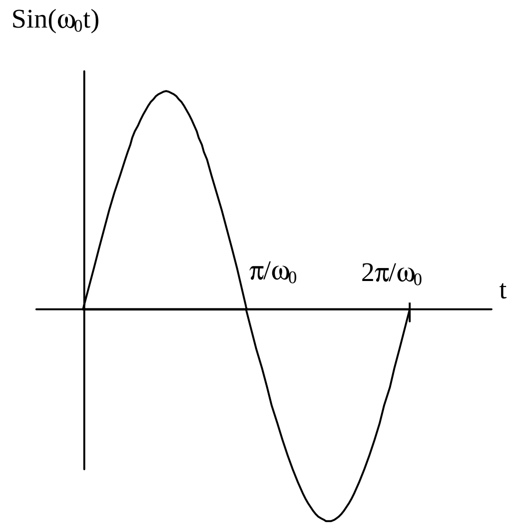
Yes, it is periodic. The period can be found by two methods. First, a trigonometric identity can be used:

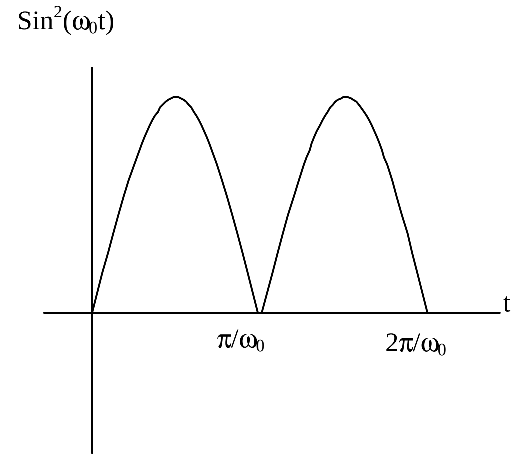
$$\sin^2(\omega_0 t) = \frac{1 - \cos(2\omega_0 t)}{2}$$

The frequency of this signal is $\omega_1 = 2\omega_0$. The period of the signal is

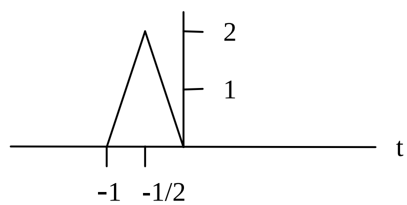
$$T_1 = \frac{2\pi}{\omega_1} = \frac{\pi}{\omega_0} = \frac{T_0}{2}.$$

The period can also be found graphically:





(c)Write the following signal in terms of a weighted sum of a combination of one or all of the
following functions: δ(t), u(t), r(t).



$$x(t) = 4r(t+1) - 8r(t+\frac{1}{2}) + 4r(t)$$

(d) Compute the energy value of the signal in (c).

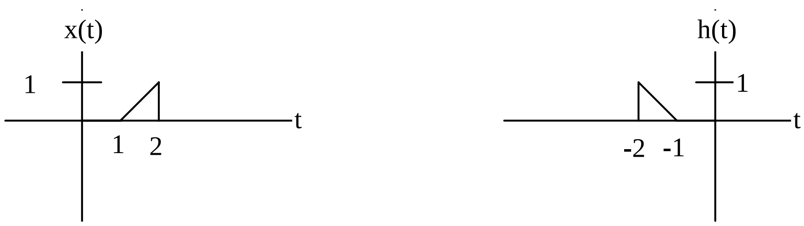
To simplify the integral, we can shift the signal to the right one unit and integrate it from zero to ½ and multiply the result by two. This is due to the symmetry associated with the signal.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$E = 2\int_{0}^{\frac{1}{2}} (4t)^2 dt = 32\frac{t^3}{3}\Big|_0^{\frac{1}{2}} = \frac{4}{3}\ Joules$$

Problem No. 2: Time-Domain Solutions

Consider the signal and system (completely described by its impulse response):



(a) Compute and plot output, y(t), for the system shown above.

The output will be the convolution of the signals x(t) and h(t). For the computation, the convolution integral will be used:

$$x(t)*h(t)=\int x(t-\lambda)h(\lambda)d\lambda$$

First, equations must be written for x and h in terms of t:

$$x(t)=\left\{\begin{matrix} t-1, 1\le t\le 2 \\ 0, otherwise \end{matrix}\right\}$$

$$h(t)=\left\{\begin{matrix} -(t+1), -2\le t\le -1 \\ 0, otherwise \end{matrix}\right\}$$

To fit the convolution integral, x must be written in terms of (t-λ):

$$x(t-\lambda)=\begin{cases} (t-\lambda-1), t-1\ge\lambda\ge t-2 \\ 0, otherwise \end{cases}$$

The convolution integral will be zero for all t except between -1 and 1. The integral will be split into two separate integrals over the ranges of t from -1 to 0 and from 0 to 1.

1. $-\int_{-2}^{t-1}[(t-\lambda-1)(\lambda+1)d\lambda], -1\le t\le 0$

2. $-\int_{t-2}^{-1}[(t-\lambda-1)(\lambda+1)d\lambda], 0\le t\le 1$

1.

$$\int_{-2}^{t-1}[\lambda^2 - \lambda t - t + 2\lambda + 1]d\lambda = [\frac{\lambda^3}{3} - t\frac{\lambda^2}{2} - t\lambda + \lambda^2 + \lambda]\Big|_{-2}^{t-1}$$

$$= [\frac{1}{3}(t-1)^3 - \frac{1}{2}t(t-1)^2 - t(t-1) + (t-1)^2 + t - 1] - [-\frac{8}{3} - 2t + 2t + 4 - 2]$$

$$= -\frac{1}{6}t^3 + \frac{1}{2}t + \frac{1}{3}, for(-1 \le t \le 0)$$

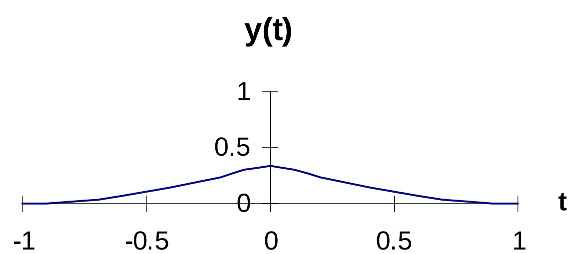
2. This is the same integral as above, but evaluated at different points:

$$[\frac{\lambda^3}{3} - t\frac{\lambda^2}{2} - t\lambda + \lambda^2 + \lambda]\Big|_{t-2}^{-1}$$

$$= [-\frac{1}{3} - \frac{1}{2}t + t + 1 - 1] - [\frac{1}{3}(t-2)^3 - \frac{1}{2}t(t-2)^2 - t(t-2) + (t-2)^2 + t - 2]$$

$$= \frac{1}{6}t^3 - \frac{1}{2}t + \frac{1}{3}, for(0 \le t \le 1)$$

The maximum value of the output occurs at t=0 and is equal to 1/3. A plot of the output is shown below:



(b) Without using the answer to part (a), explain whether the system is causal.

A necessary condition for causality is that $h(t)=0$ for $t<0$. However, the plot of $h(t)$ clearly shows that it starts prior to $t=0$, so the system is noncausal.

(c) Use your answer to part (a) to support your reasoning given in (b).

The output $y(t)$ begins prior in time to the input $x(t)$, so the system must be noncausal.

(d) Describe the system using as many system modeling concepts as possible (for example, linearity). Justify your answers.

1,2. The system is both linear and time-invariant because it is completely described by its impulse response.

1. Linear: $\alpha_1 H\{x_1(t)\} + \alpha_2 H\{x_2(t)\} = H\{\alpha_1 x_1(t)+\alpha_2 x_2(t)\}$
2. Time Invariant: $H\{x(t-T)\}=y(t-T)$
3. Noncausal (for reasons stated above)
4. Bounded-input bounded-output stable: Every bounded input results in a bounded output. Also, to be BIBO stable:

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

$h(t)_{shifted} = -t \qquad -1 \le t \le 0$

$$\int_{-1}^{0} t dt = \frac{t^2}{2}\Big|_{-1}^{0} = \frac{1}{2}$$

5. Continuous Time System: signals are continuous through time, they do not occur only at discrete locations.

Problem No. 3: Fourier Series

Given the signal $x(t) = 3\sin(1.5\omega_1 t) + 5\cos(1.75\omega_1 t)$,

(a) Using symmetry arguments, explain which Fourier coefficients of the trigonometric Fourier
Series will be zero ({an} and {bn}). Be careful and be precise.

x(t) is the sum of an odd function (sine) and an even function (cosine). This summation results in no symmetry. Since there is no symmetry, we cannot state that either {an} or {bn} will be zero.

(b) Compute the Fourier series coefficients.

The first step is to find out if x(t) is periodic. We look for a period common to both the sine and cosine waves:

$n_A\omega_0 = 1.5\omega_1$
$n_B\omega_0 = 1.75\omega_1$

Dividing the first equation by the second, we get:

$$\frac{n_A}{n_B} = \frac{1.5}{1.75} = \frac{6}{7}$$

$6\omega_0 = 1.5\omega_1$ or $\omega_0 = \omega_1/4$.

We can use this to solve for T:

$T = 8\pi/\omega_1$ or $\omega_1 = 8\pi/T$

$$a_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{T}\int_0^T 3\sin(1.5(\frac{8\pi}{T})t)dt + \frac{1}{T}\int_0^T 5\cos(1.75(\frac{8\pi}{T})t)dt$$

$$a_0 = -\frac{3}{T}(\frac{T}{1.5(8\pi)})\left[\cos(1.5(8\pi)) - 1\right] + \frac{5}{T}\frac{T}{1.75(8\pi)}\left[\sin(1.75(8\pi))\right]$$

$$a_0 = 0$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)dt$$

$$= \frac{6}{T}\int_0^T \sin\left(\frac{1.5(8\pi)}{T}t\right)\cos\left(n\frac{2\pi}{T}t\right)dt + \frac{10}{T}\int_0^T \cos\left(\frac{1.75(8\pi)}{T}t\right)\cos\left(n\frac{2\pi}{T}t\right)dt$$

The following integrals will be used to solve for {an} and {bn}:

$$\int \sin(mu)\cos(nu)du = -\frac{\cos(m-n)u}{2(m-n)} - \frac{\cos(m+n)u}{2(m+n)}$$

$$\int \cos(mu)\cos(nu)du = \frac{\sin(m-n)u}{2(m-n)} + \frac{\sin(m+n)u}{2(m+n)u}$$

$$a_n = 3\left[-\frac{\cos(1.5(8\pi)-n2\pi)t}{1.5(8\pi)-n2\pi} - \frac{\cos(1.5(8\pi)+n2\pi)t}{1.5(8\pi)-n2\pi} + \frac{1}{1.5(8\pi)-n2\pi} + \frac{1}{1.5(8\pi)+n2\pi}\right]$$

$$+5\left[\frac{\sin(1.75(8\pi)-n2\pi)}{1.75(8\pi)-n2\pi} + \frac{\sin(1.75(8\pi)+n2\pi)}{1.75(8\pi)+n2\pi)}\right]$$

Every {an} term can be shown to equal zero for all n except n = 7. When n=7:

$$a_n = 5\left[\frac{-2\pi\cos(0)}{-2\pi}\right] = 5$$

$$a_7 = 5$$

(Done using L'Hopital's Rule)

Similarly,

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)dt$$

$$= \frac{6}{T}\int_0^T \sin\left(\frac{1.5(8\pi)t}{T}\right)\sin\left(n\frac{2\pi}{T}t\right)dt + \frac{10}{T}\int_0^T \cos\left(\frac{1.75(8\pi)t}{T}\right)\sin\left(n\frac{2\pi}{T}t\right)dt$$

$$= 3\left[\frac{\sin(1.5(8\pi)-n2\pi)}{1.5(8\pi)-n2\pi} - \frac{\sin(1.5(8\pi)+n2\pi)}{1.5(8\pi)+n2\pi)}\right] + 5\left[-\frac{\cos(n2\pi-1.75(8\pi))}{n2\pi-1.75(8\pi)} - \frac{\cos(n2\pi+1.75(8\pi))}{n2\pi+1.75(8\pi)}\right]$$

$$-5\left[-\frac{1}{n2\pi-1.75(8\pi)} - \frac{1}{n2\pi+1.75(8\pi)}\right]$$

{bn} terms are equal to zero for all n except for n = 6. When n = 6:

$$b_n = 3\left[\frac{-2\pi\cos(0)}{-2\pi}\right]$$

$$b_6 = 3$$

(Done using L'Hopital's Rule)