Name: Tim Schlichter

Problem	Points	Score
1a	10	
1b	10	
1c	10	
1d	10	
2a	10	
2b	10	
2c	10	
3a	10	
3b	10	
3c	10	
Total	100	

Notes:

1. The exam is closed books/closed notes - except for one page of notes.

2. Please show ALL work. Incorrect answers with no supporting explanations or work will be given no partial credit.

3. Please indicate clearly your answer to the problem.

Problem No. 1: For the linear time-invariant system: $H(s) = \dfrac{s+2}{(s^2 - 2s - 3)}$

(a) Find the state variable description of the system.

$$H(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$$

$$A = (s-3)H(s)\Big|_{s=3} = \frac{5}{4}$$

$$B = (s+1)H(s)\Big|_{s=-1} = -\frac{1}{4}$$

$$Y(s) = \frac{5}{4}\left[\frac{U(s)}{s-3}\right] - \frac{1}{4}\left[\frac{U(s)}{s+1}\right]$$

Now, we will define X_1 and X_2.

$$X_1(s) = \frac{U(s)}{s-3}$$

$$X_2(s) = \frac{U(s)}{s+1}$$

$$Y(s) = \frac{5}{4}X_1(s) - \frac{1}{4}X_2(s)$$

By taking the inverse Laplace transform, the time domain expression, $y(t)$, is found:

$$y(t) = \frac{5}{4}x_1(t) - \frac{1}{4}x_2(t)$$

$$x_1(t) = e^{3t}u(t)$$

$$x_2(t) = e^{-t}u(t)$$

$$\dot{x}_1 = 3x_1 + u$$

$$\dot{x}_2 = -x_2 + u$$

Thus, the state variable description in matrix form is given as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u$$

$$y = \begin{bmatrix} \frac{5}{4} & -1/4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b) Compute the state transition matrix, $\phi(t)$.

$$\Phi(s) = (sI - A)^{-1}$$

$$(sI - A) = \begin{bmatrix} s-3 & 0 \\ 0 & s+1 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{(s-3)(s+1)} \begin{bmatrix} s+1 & 0 \\ 0 & s-3 \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s-3} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

$$\phi(t) = L^{-1}\{\Phi(s)\} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} u(t)$$

(c) Using the state variable representation, implement this as an RLC circuit.

Due to the e^{3t} expression, the system is unstable. Therefore, it is impossible to represent an unstable system using an RLC circuit. All RLC circuits are stable.

Problem No. 2: This problem deals with various aspects of Z-Transforms.

(a) Derive the expression for the Z-Transform of $x(n) = na^{-n}u(n)$.

We will start with entry 3 of Table 8-1:

$$\sum_{n=0}^{\infty} e^{-an} z^{-n} = \frac{1}{1 - e^{-a} z^{-1}}$$

Since, e^a is a constant, we will let $e^a = a$ to fit the expression given in the problem:

$$\sum_{n=0}^{\infty} a^{-n} z^{-n} = \frac{1}{1 - a^{-1} z^{-1}}$$

Now, differentiating both sides with respect to z:

$$\sum_{n=0}^{\infty} a^{-n} (-n) z^{-n-1} = \frac{-a^{-1} z^{-2}}{1 - a^{-1} z^{-1}}$$

Multiplying both sides by -z:

$$\sum_{n=0}^{\infty} n a^{-n} z^{-n} = \frac{a z^{-1}}{(1 - a^{-1} z^{-1})^2}$$

$$X(z) = \frac{a z^{-1}}{(1 - a^{-1} z^{-1})^2}$$

(b) For the transfer function, $H(z) = \dfrac{1 - (1/2) z^{-1}}{1 - (3/2) z^{-1} - z^{-2}}$, find a closed-form expression for $h(n)$ (don't use long division).

$$H(z) = \frac{\frac{z - 0.5}{z}}{\frac{z^2 - 1.5z - 1}{z^2}} = \frac{z(z - 0.5)}{(z^2 - 1.5z - 1)}$$

$$\frac{H(z)}{z} = \frac{z - 0.5}{(z + 0.5)(z - 2)} = \frac{A}{z + 0.5} + \frac{B}{z - 2}$$

$$A = \frac{2}{5}$$

$$B = \frac{3}{5}$$

$$H(z) = \frac{2}{5} \frac{z}{z + 0.5} + \frac{3}{5} \frac{z}{z - 2}$$

$$h(n) = \left[\frac{2}{5} \left(-\frac{1}{2} \right)^n + \frac{3}{5} (2)^n \right] u(n)$$

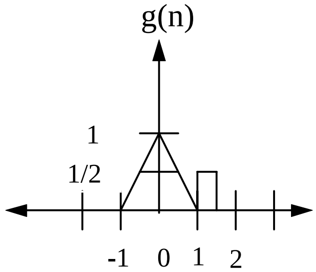
(c) Is the system stable?

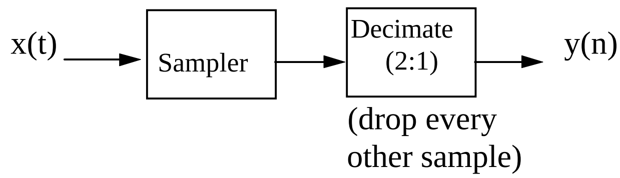
For stability, the poles must be inside the unit circle in the z-plane. Since one of the poles is at 2, it is clearly outside of the unit circle, and the system is unstable.

(d) Is the system causal? Explain.

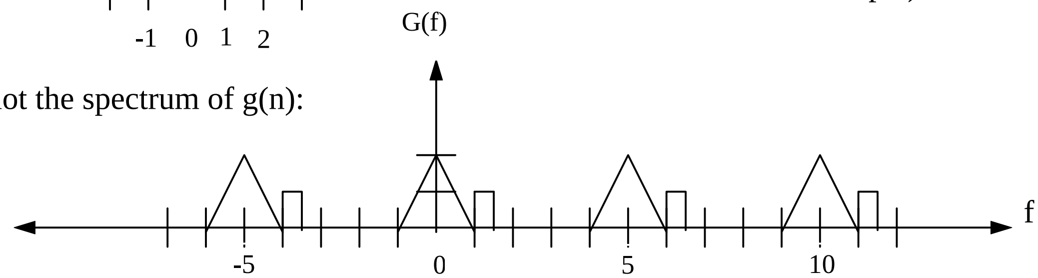
Yes, the system is causal. This is obvious because the transfer function has a u(n) term in it. This indicates that the transfer function begins at t=0. Thus, the output cannot begin before t=0, and the system is causal.

Problem No. 3: For the system shown:

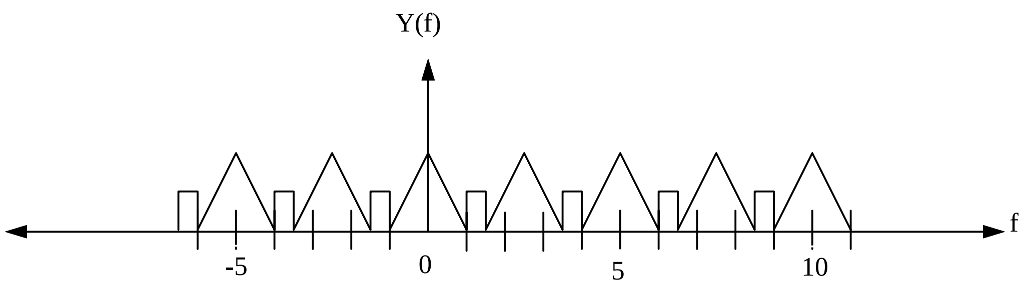




(a) Plot the spectrum of g(n):



(b) Plot the spectrum of y(n):



(c) How do you explain the fact that the sample frequency of y(n) is less than the Nyquist rate, yet there is no distortion?

The Nyquist Rate is 3Hz. This is found by seeing that the original signal's spectrum, X(f), has a width of 1.5Hz (where the width is measured from the origin to the end of the signal). The Nyquist Rate is twice that width, or 3Hz. The sample frequency of y(n) is 2.5Hz due to the decimator. We would expect distortion since the sample frequency is less than the Nyquist Rate. However, the signal is not symmetric about the y axis due to the small pulse after the triangle function. The width on the left side of the y axis is 1Hz, as opposed to 1.5Hz to the right of the y axis. Therefore, when the spectrum of y(n) is constructed, there is no distortion and the samples line up one right after another. This problem indicates that the method we have learned for finding the Nyquist rate is not as precise as it could be. The Nyquist rate found using our method is higher than what it actually could be.