Name:

Problem	Points	Score
1a	10	
1b	10	
1c	10	
2a	10	
2b	10	
2c	10	
2d	10	
3a	10	
3b	10	
3c	10	
Total	100	

Notes:

1. The exam is closed books/closed notes - except for one page of notes.

2. Please show ALL work. Incorrect answers with no supporting explanations or work will be given no partial credit.

3. Please indicate clearly your answer to the problem. If I can't read it (and I am the judge of legibility), it is wrong. If I can't follow your solution (and I get lost easily), it is wrong. All things being equal, neat and legible work will get the higher grade:)

Problem No. 1: Fourier Series

(a) For the system below, compute the Fourier series of the output.

$$x(t) = A\cos(\omega_o t) \longrightarrow \boxed{y(t) = x^2(t)} \longrightarrow y(t) = ???$$

The formal definition of the trignometric Fourier series is

$$f(t) = a_o + \sum_{n=1}^{\infty} a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t)$$

where, $a_o = \frac{1}{T_o}\int_{T_o} x(t)dt \qquad a_n = \frac{2}{T_o}\int_{T_o} [x(t)\cos(\omega_o t)]dt \qquad b_n = \frac{2}{T_o}\int_{T_o} [x(t)\sin(\omega_o t)]dt$

Yet, by using trig identities and knowledge about the cosine function, the fourier series can be obtained without calculating the integrals.

$$y(t) = x^2(t) = (A\cos\omega_o t)^2$$

$$y(t) = A^2\cos^2\omega_o t$$

use the trignometric identity: $\cos^2 u = \frac{1}{2}(1+\cos 2u)$

after applying the identity

$$y(t) = A^2\left[\tfrac{1}{2}(1+\cos 2\omega_o t)\right]$$

and simplifying

$$y(t) = A^2/2 + (A^2/2)(\cos 2\omega_o t)$$

Since the cosine function is even, $b_n = 0$ & $a_n \neq 0$; therefore there are only a_n terms in the in the fourier series. Also, the fourier series of the cosine function is the cosine function. Using the above information about the cosine function and about what y(t) is we obtain the following for the coefficients of the fourier series.

$$a_0 = A^2/2 \qquad\qquad a_2 = A^2/2$$

So the fourier series is $y(t) = a_o + a_2\cos 2\omega_o t$

substitute in the values for a_o and a_2 to get the complete fourier series,

$$y(t) = A^2/2 + (A^2/2)(\cos 2\omega_o t)$$

(b) Compute the energy and power of the output. (Comment: Think about orthogonality before you write the answer that is most obvious.)

$y(t) = A^2 / 2 + A^2 / 2 \cos^2 w_o t$

Since the output is a periodic function with infinite area under the curve its energy is infinite, E = ∞. This can be proven by putting the function y(t) into the definition of energy.

$$E = \lim_{T_o \to \infty} \int_{-T_o}^{T_o} [y(t)]^2 \, dt$$

The power needs to be calculated since periodic functions are power signals.

$$P = \lim_{T_o \to \infty} \frac{1}{2T_o} \int_{-T_o}^{T_o} [y(t)]^2 \, dt = \lim_{T_o \to \infty} \frac{1}{2T_o} \int_{-T_o}^{T_o} \left[A^2/2 + \left(A^2/2\right)(\cos 2\omega_o t)\right]^2 dt$$

$$P = \lim_{T_o \to \infty} \frac{1}{2T_o} \int_{-T_o}^{T_o} \left[A^4/4 + A^4 \cos 2\omega_o t + \left(A^4/4\right)\cos^2 2\omega_o t\right] dt$$

substitute $\omega_o = 2\Pi/T_o$ into the functions and integrate. The first cosine integral will go to zero because $\sin(4\Pi) = 0$. Now we are left with

$$P = \lim_{T_o \to \infty} A^4/4 + \frac{1}{2T_o} \int_{-T_o}^{T_o} \left[\left(A^4/4\right)\cos^2 2\omega_o t\right] dt$$

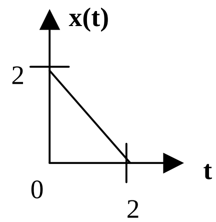
using the trignometric identity $\cos^2 u = \tfrac{1}{2}(1 + \cos 2u)$

$$P = \lim_{T_o \to \infty} A^4/4 + \left(\frac{A^4}{8T_o}\right) \int_{-T_o}^{T_o} \left[\tfrac{1}{2}(1 + \cos 4\omega_o t)\right] dt = \lim_{T_o \to \infty} A^4/4 + \left(\frac{A^4}{8T_o}\right) \int_{-T_o}^{T_o} \frac{1}{2} dt + \left(\frac{A^4}{8T_o}\right) \int_{-T_o}^{T_o} \left(\tfrac{1}{2}\cos 4\omega_o t\right) dt$$

For the same reasons above, the cosine term will go to zero.

$$P = \lim_{T_o \to \infty} A^4/4 + A^4/8 = A^4/4 + A^4/8$$

(c) Now assume x(t) is as shown to the right. Compute the energy and power of y(t).



$$x(t) = -t + 2 \qquad 0 \le t \le 2$$

$$y(t) = x^2(t) = (-t+2)^2$$

Calculate the energy.

$$E = \lim_{T_o \to \infty} \int_{-T_o}^{T_0} \left[(-t+2)^2\right]^2 dt = \int_0^2 (-t+2)^4\, dt$$

Substitute $u = -t + 2$
$du = -dt$

$$E = \int (u)^4 (-du) = \int -u^4 du = -\frac{u^5}{5}$$

Substitute back to and evaluate.

$$E = -\left.\frac{(-t+2)^5}{5}\right|_0^2 = \left(-\frac{1}{5}\right)(0-32) = \frac{32}{5} = 6.4\,Joules$$

Calculate the power.

$$P = \lim_{T_o \to \infty} \frac{1}{2T_o} \int_{-T_o}^{T_o} \left[(-t+2)^2\right]^2 dt = \lim_{T_o \to \infty} \frac{1}{2T_o} \int_{-T_o}^{T_o} (-t+2)^4 dt$$

Subsitiuting as done for the energy.

$$P = \lim_{T_o \to \infty} \frac{1}{2T_o} \int (u)^4 (-du) = \lim_{T_o \to \infty} \frac{1}{2T_o} \int -u^4 du = \lim_{T_o \to \infty} \left(\frac{1}{2T_o}\right)\left(-\frac{u^5}{5}\right)$$

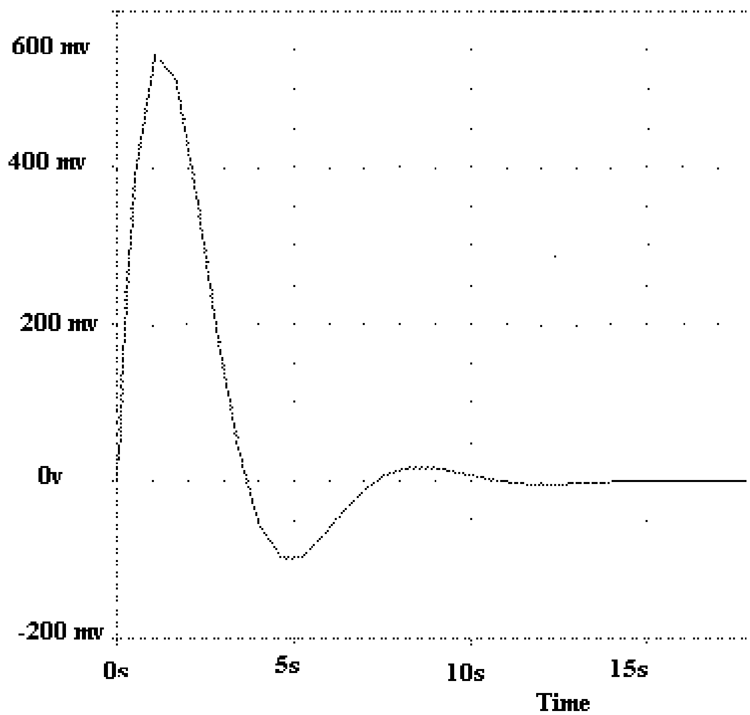
Substitiute back and evaluate.

$$P = \lim_{T_o \to \infty} \left(\frac{1}{2T_o}\right)\left(-\frac{(-t+2)^5}{5}\right)\Bigg|_0^2 = \lim_{T_o \to \infty} \left(\frac{1}{2T_o}\right)\left(-\frac{1}{5}\right)(0-32) = \lim_{T_o \to \infty} \left(\frac{1}{2T_o}\right)\left(\frac{32}{5}\right) = 0$$

Problem No. 2: Time-Domain Solutions

(a) For the circuit shown, using concepts developed in this class, sketch the output signal, , if x(t) is the unit step function. Explain.

$$v(t) = e^{-\alpha t}(\beta_1 \cos(\omega_d t) + \beta_2 \sin(\omega_d t))$$



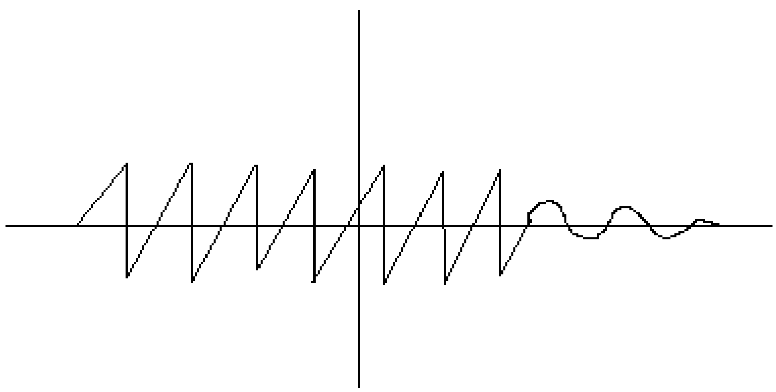
The graph above is the natural response of the series RLC circuit in this problem. The unit step input function produces this type of output because the unit step function mimics the natural operation of the circuit. The circuit is only on for time t>0. If a source such as a battery was connected to the circuit, then current would follow until the battery was drained. Although the battery has a finite operating time, the battery would produce the same input as the unit step function. The above waveform is called the underdamped response. The waveform occures because as the capacitor charges, it causes the current in the circuit to go to zero. When the capacitor is fully charged, it forces the voltage across the rest of the circuit to change polarity. Eventually current flows as the votlage changes. This makes the waveform dip below the zero volt mark. Once the capacitor is discharged, it will begin to charge again, but much of the power supplied by the input has been used, so the capacitor charges to a lower value. The cycle continues until all the stored power in the input is used and the waveform reaches zero volts.

(b) Sketch the output if x(t) is a unit pulse with width of 1 millsecond. Explain.

The output for x(t) as a unit pulse would have some of the same characteristics as the unit step response.   The output would start before zero since the unit pulse function starts before t=0. The waveform would look like an underdamped sinusoid, but the amplitude of the waveform would be greatly reduced. The amplitude would be reduced because the convolution of the two functions, the underdamped sinusoid and the unit pulse, would have a smaller area than the convolution of the underdamped sinusoid with the unit step fuinction.

(c) Sketch the output if x(t) is a periodic sequence of unit pulses with a period of 10 seconds.

The output of the pulse train would look like a saw-tooth function. As one pulse passed part of the underdamped sinusoid, another would then follow. The waveforms would then overlap causing sharp ramps and quick drops in amplitude. As the last pulse crossed the input waveform, the output would look more like the underdamped sinusoid's waveform.



(d) Suppose the period is dropped to 2 seconds. How does this affect the shape of the output?

Reducing the period from 10 seconds to 2 seconds will increase the frequency. This will cause more waveforms to appear in a given time period. The waveform would be similar to the above waveform except the peaks and troughs would be closer together causing increased overlap.

Problem No. 3: The dreaded thought problem...

(a) Using concepts developed in the first three chapters of the book, characterize the cruise control circuit in an automobile as a system.

- The cruise control of a car is a low pass filter because it cuts off all high frequency inputs to create a smooth steady change of speed.
- The system is causal because it does not anticipate it's future input. It is not possible for the system to anticipate the terrain it encounters because the does not know the path.
- The system is also instantaneous because it does not use past values of the input to create an output.
- The system is linear because when you increase the speed of the car or when it receives an input you get the output to be the speed of the car plus the increase. Therefore, this is a linear relationship in the fact that superposition holds.
- The system is fixed because the input output relationship does not change with time.

(b) If this system were a linear time-invariant system, design the impulse response of a "good" system. Explain how you might implement this in a circuit.

Since cruise control acts like a low pass filter, it can be modified to into an integrator circuit that would smooth out the discontinuities obtained in the input from bumps and other road anomolies.
This could be implemented by using an RC circuit.

(c) It is well-known that cruise controls are not optimal for terrain that is mountainous (such as ski country in Colorado). Why (explain this in terms of concepts discussed in this class)? What modifications would you need to make (again, explain this in terms of concepts such as causality)?

A cruise control system is not optimal for mountaineous terrian because of the vast changes in the path (steepness of inclines). The problem revolves around the fact that the cruise control system is causal. To make the system better, it would need to know what the terrain looked like before it reached a particluar spot. Therefore, the system would need either a way to project what the terrain would look like to produce a good output or the system would need a pre-programed map of the inputs for a given region to produce useful outputs.