Name:

Problem	Points	Score
1a	10	
1b	10	
1c	10	
1d	10	
2a	10	
2b	10	
2c	10	
2d	10	
3a	10	
3b	10	
Total	100	

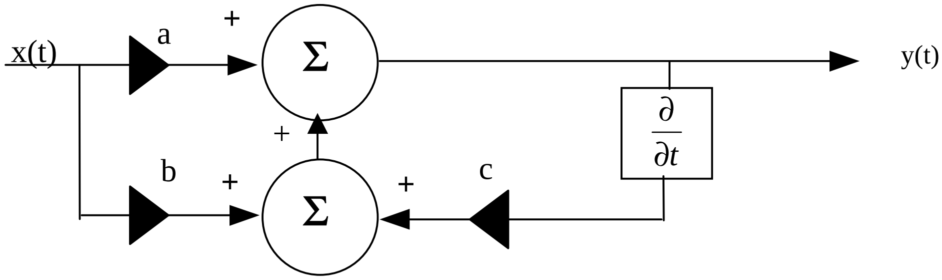
Notes:

1. The exam is closed books/closed notes - except for one page of notes.

2. Please show ALL work. Incorrect answers with no supporting explanations or work will be given no partial credit.

3. Please indicate clearly your answer to the problem. If I can't read it (and I am the judge of legibility), it is wrong. If I can't follow your solution (and I get lost easily), it is wrong. All things being equal, neat and legible work will get the higher grade:)

Problem No. 1: Block Diagrams



(a) Write a differential equation describing this system.

The output from the "a" amplifier is ax(t) and the output from the "b" amplifier is bx(t). The output of the "c" amplifier is $c\frac{\partial}{\partial t}y(t)$. Putting these pieces together yields a differential equation for the system.

$$y = ax + bx + c\frac{\partial y}{\partial t}$$

Which can be rearranged as

$$c\frac{\partial y}{\partial t} - y + ax + bx = 0\,.$$

(b) Find the transfer function.

To obtain the transfer function of the above differential equation, it must be converted to the frequency domain using Laplace transforms. Using the Laplace transforms below,

$$\mathcal{L}\left\{\frac{\partial y(t)}{\partial t}\right\} = sY(s) \qquad \mathcal{L}\{y(t)\} = Y(s) \qquad \mathcal{L}\{x(s)\} = X(s)$$

the differential equation transforms into $csY(s) - Y(s) + aX(s) + bX(s) = 0$ which can be rewritten as

$$(cs-1)Y(s) = -(a+b)X(s)\,.$$

The transfer function is defined as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{a+b}{1-sc} = \frac{\frac{-(a+b)}{c}}{s-\frac{1}{c}}$$

(c) For what values of a, b, and c is the system stable (consider only non-zero values of a, b, and c).

The system is stable when c<0. When this occurs, the poles of the transfer function are located in the left half plane therefore the system is stable.
When c>0, the poles are located in the right half plane causing the system to be unstable. This can be readily modeled by using the java spectrum analysis tool or by looking at the transfer function in the time domian.
Since a and b are defined as non-zero and are constants, they do not affect the stability of the system.

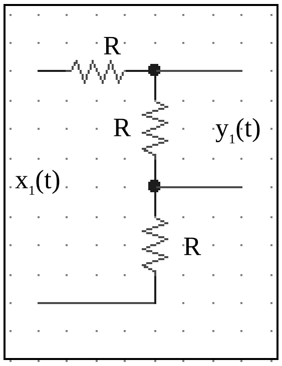
(d) Find the impulse response.

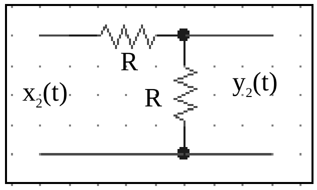
The impluse response of the system is found by taking the inverse Laplace transform of the transfer function.

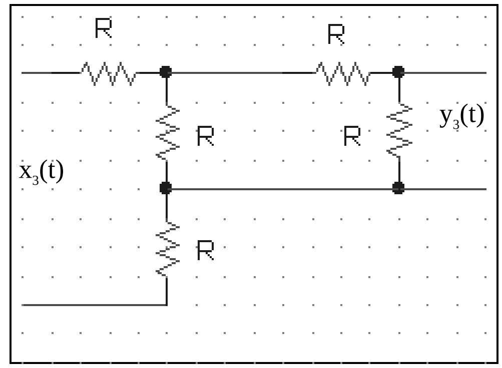
$$\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{\frac{-(a+b)}{c}}{s-\frac{1}{c}}\right\} = \frac{-(a+b)}{c} e^{\frac{1}{c}t} u(t)$$

Problem No. 2: Transfer Functions

For the circuit shown below:



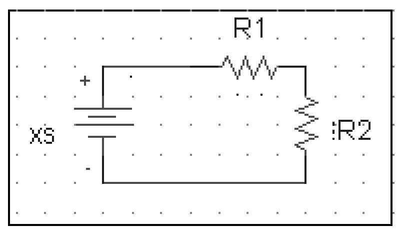




(a) Find $H_1(s)$:

The system defined by $H_1(s)$ is a voltage divider. The formula for a voltage divider is

$x_1 = \frac{R_1}{R_1 + R_2} x_s$ for a circuit of this type



Transform the system to the frequency domain by using Laplace transforms

$$\mathcal{L}\{y(t)\} = Y(s) \qquad \mathcal{L}\{x(s)\} = X(s)$$

so that

$$Y_1(s) = \frac{R}{R+R+R} X_1(s) = \frac{R}{3R} X_1(s) = \frac{1}{3} X_1(s)$$

and the transfer function is

$$H_1(s) = \frac{Y_1(s)}{X_1(s)} = \frac{1}{3}$$

(b) Find $H_2(s)$:

This system is a voltage divider like system 1, therefore, placing the system in the frequency domain produces the equation

$$Y_2(s) = \frac{R}{R+R} X_2(s) = \frac{R}{2R} X_2(s) = \frac{1}{2} X_2(s)$$

This gives the transfer function

$$H_2(s) = \frac{Y_2(s)}{X_2(s)} = \frac{1}{2}$$

(c) Find $H_3(s)$:

Although this system is a combination of the two previous systems, the equation of the system is not simply a voltage divider. Instead, more advanced circuit analysis is needed to find y(t). Since the transfer function is required, it is simplier to solve the circuit in the frequency domain .

First, find the voltage across the middle resistor of the 3 resistors in series.

$$Z(s) = \frac{R\|2R}{2R + R\|2R} X_3(s) = \frac{\frac{2}{3}R}{2R + \frac{2}{3}R} X_3(s) = \frac{\frac{2}{3}R}{\frac{8}{3}R} X_3(s) = \frac{1}{4} X_3(s)$$

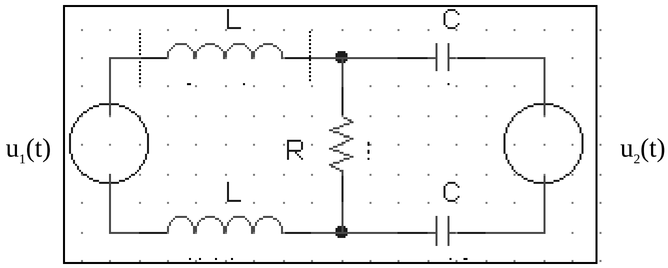
Next find $Y_2(s)$ by using the fact that the voltage across paths in parallel is equal and that the transfer function of the second half of the circuit is ½.

$$Y_3(s) = Z(s) * \frac{1}{2} = \frac{1}{4} X_3(s) * \frac{1}{2} = \frac{1}{8} X_3(s)$$

(d) Is $H_3(s) = H_1(s) \bullet H_2(s)$? Justify your answer. Use as many concepts developed in this course as possible. A yes/no answer with no explanation gets no credit.

The product of the first two transfer functions is not equal to the third transfer function. This is true because the third circuit experiences loading since there are circuit elements connected to the middle resistor in parallel.

Problem No. 3: The “Interesting” Problem



(a) Assume the voltage across the resistor in the circuit above is the output voltage, y(t). Derive the state variables representation of this circuit.

The state variable representation of a circuit is defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ and } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

To solve this circuit, write KVL equations for the two meshes.

For the first mesh: $-u_1(t) + L\dot{i}_1 + R(i_1 - i_2) + L\dot{i}_1 = 0$

$$-u_1(t) + 2L\dot{i}_1 + Ri_1 - Ri_2 = 0 \qquad (1)$$

For the second mesh: $u_2(t) + v_c + R(i_2 - i_1) + v_c = 0$

$$u_2(t) + 2v_c + Ri_2 - Ri_1 = 0 \qquad (2)$$

Next define the state variables and the two mesh currents.

$$x_1 = v_c \qquad i_1 = i_L$$
$$x_2 = i_L \qquad i_2 = C\dot{v}_c$$

Susbstitute the state variables into equations (1) and (2).

For equation (1): $-u_1(t) + 2L\dot{i}_L + Ri_L - RC\dot{v}_c = 0$

$$\dot{i}_L(2L) = RC\dot{v}_c + u_1(t) \qquad (3)$$

For equation (2): $u_2(t) + 2v_c + RC\dot{v}_c - Ri_L = 0$

$$RC\dot{v}_c = -2v_c + Ri_L - u_2(t) \qquad (4)$$

Substitute equation (4) into equation (3) to remove the $\dot{v}_c$ componet from equation (3).

$$\dot{i}_L(2L) = -2v_c + Ri_L - u_2(t) + u_1(t)$$

Now the state equations can be obtained.

$$\begin{bmatrix} \dot{v}_c \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} \frac{-2}{RC} & \frac{1}{C} \\ \frac{-1}{L} & \frac{R}{2L} \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} 0 & \frac{-1}{RC} \\ \frac{1}{2L} & \frac{-1}{2L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Next we need to obtain the solution equations. Since there is only one output, the matrix will contain only one variable.

$$y(t) = R(i_1 - i_2)$$

Substitute in the state variables to get $y(t) = Ri_L - RC\dot{v}_c$.
Next substitute in equation (4) to get rid of the derivative term $\dot{v}_c$.

$$y(t) = Ri_L - (-2v_c + Ri_L - u_2(t))$$
$$y(t) = 2v_c + u_2(t)$$

Now we put this into matrix form.

$$[y(t)] = [2 \quad 0] \begin{bmatrix} v_c \\ i_L \end{bmatrix} + [0 \quad 1] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

(b) Now that you have finished (a), tell me how many state variables you should have used, and identify which voltages and/or currents they correspond to :)

This system should be solved using 2 state variables. The current in the two inductors is identical since the inductors are the same. The capacitor vaoltages are also the same because the capacitors have the same value. If four state variables had been used, one for each inductor current and capacitor voltage, the same solution would be reached as using two state variables. In the four by four matrix some of the terms would cancel producing the same results.