Name: Reggie Warren

Problem	Points	Score
1a	10	
1b	10	
1c	10	
1d	10	
2a	10	
2b	10	
2c	10	
2d	10	
3a	10	
3b	10	
Total	100	

Notes:

1. The exam is closed books/closed notes - except for one page of notes.

2. Please show ALL work. Incorrect answers with no supporting explanations or work will be given no partial credit.

3. Please indicate clearly your answer to the problem. If I can't read it (and I am the judge of legibility), it is wrong. If I can't follow your solution (and I get lost easily), it is wrong. All things being equal, neat and legible work will get the higher grade:)

Problem No. 1: Block Diagrams

(a) Write a differential equation describing this system.

The differential equation that describes this system is found in the following manner: Collecting the terms at the output node yields:

$$y(t) = ax(t) + bx(t) + c\frac{dy(t)}{dt} \Rightarrow$$

$$c\frac{dy(t)}{dt} - y(t) = -ax(t) - bx(t) \Rightarrow$$

$$\frac{dy(t)}{dt} - \frac{1}{c}y(t) = -\frac{1}{c}ax(t) - \frac{1}{c}bx(t)$$

(b) Find the transfer function.

$$L\left\{\frac{dy(t)}{dt}\right\} - \frac{1}{c}L\{y(t)\} = -\frac{a}{c}L\{x(t)\} - \frac{b}{c}L\{x(t)\} \Rightarrow$$

$$sY(s) - sy(0^-) - \frac{1}{c}Y(s) = -\frac{a}{c}X(s) - \frac{b}{c}X(s)$$

Since we already have the equation which characterizes the circuit, the transfer function can be found by first taking the equation's Laplace transform:

$$sY(s) - \frac{1}{c}Y(s) = -\frac{a}{c}X(s) - \frac{b}{c}X(s) \Rightarrow$$

$$Y(s)\left[s - \frac{1}{c}\right] = X(s)\left[\frac{-(a+b)}{c}\right] \Rightarrow$$

$$\frac{Y(s)}{X(s)} = \frac{-(a+b)}{sc-1}$$

Assuming zero initial conditions:

(c) For what values of a, b, and c is the system stable (consider only non-zero values of a, b, and c).

Stability of a system implies that there are no poles in the right-half plane. Mathematically speaking, the denominator of the transfer function has to be of the form $s + \alpha$, which would imply via the inverse Laplace transform that the transfer function's time-domain equivalent is a decaying exponential, a function whose integral for all time converges, implying stability. Therefore, given that *a, b,* and *c* must all be nonzero, any further restriction lies solely on *c*, and it is that *c* must be strictly nonzero.

(d) Find the impulse response.

Setting *X(s)* = 1 gives:

$$Y(s) = \frac{-(a+b)}{sc-1} \Rightarrow$$

$$Y(s) = \frac{-a}{sc-1} - \frac{b}{sc-1}$$

The impulse response, y(t), is then given as

$$y(t) = L^{-1}\{Y(s)\} - aL^{-1}\left\{\frac{1}{sc-1}\right\} - bL^{-1}\left\{\frac{1}{sc-1}\right\} \Rightarrow$$

$$y(t) = -aL^{-1}\left\{\frac{\frac{1}{c}}{s-\frac{1}{c}}\right\} - bL^{-1}\left\{\frac{\frac{1}{c}}{s-\frac{1}{c}}\right\} \Rightarrow$$

$$y(t) = -\frac{a}{c}L^{-1}\left\{\frac{1}{s-\frac{1}{c}}\right\} - \frac{b}{c}L^{-1}\left\{\frac{1}{s-\frac{1}{c}}\right\} \Rightarrow$$

$$y(t) = \frac{-(a+b)}{c}e^{\frac{1}{c}t}$$

Problem No. 2: Transfer Functions

For the circuit shown below:

(a) Find $H_1(s)$:

The input voltage divides equivalently across the three resistors, which implies

$$y_1(t) = \frac{1}{3} x_1(t),$$

which implies

$$\frac{y_1(t)}{x_1(t)} = \frac{1}{3}$$

(b) Find $H_2(s)$:

The simple voltage divider circuit yields:

$$y_2(t) = \frac{1}{2} x_2(t) \Rightarrow$$
$$\frac{y_2(t)}{x_2(t)} = \frac{1}{2}$$

(c) Find $H_3(s)$:

The voltage across the equivalent resistance in the right mesh of the circuit is found to be ¼ by finding the equivalent resistance and then performing a voltage division. This voltage then divides equally across the series resistors in the rightmost part of the circuit to give

$$\frac{y_3(t)}{x_3(t)} = \frac{1}{8}$$

(d) Is ? Justify your answer. Use as many concepts developed in this course as possible. A yes/no answer with no explanation gets no credit.

In terms of this circuit,

$$\frac{1}{2} * \frac{1}{3} \neq \frac{1}{8}$$

This occurs because of the loading effect. If loading is not neglected, then the independent cascading that would be implied no longer holds and one subcircuit becomes the load to another, which implies that a different transfer function describing the overall circuit will be obtained.

Problem No. 3: The “Interesting” Problem

(a) Assume the voltage across the resistor in the circuit above is the output voltage, y(t). Derive the state variables representation of this circuit.

Assuming that four state variables are necessary:

$$x_1 = i_L(t) \Rightarrow \dot{x}_1 = \frac{di_L(t)}{dt}$$

$$x_2 = v_c(t) \Rightarrow \dot{x}_2 = \frac{dv_c(t)}{dt}$$

$$x_3 = i_L(t) \Rightarrow \dot{x}_3 = \frac{di_L(t)}{dt}$$

$$x_4 = v_c(t) \Rightarrow \dot{x}_4 = \frac{dv_c(t)}{dt}$$

KVL gives

$$\frac{di_L(t)}{dt} = \frac{u_1(t)}{2L} - \frac{i_L(t)R}{2L} + \frac{i_2(t)R}{2L}$$

for the left mesh, where

$$i_2(t) = \frac{dv_c(t)}{dt}$$

and it gives

$$\frac{dv_c(t)}{dt} = \frac{i_L(t)}{C} - \frac{2v_c(t)R}{RC} - \frac{u_2(t)}{RC}$$

for the right mesh.

Substituting the equivalence for $i_2(t)$ into the KVL equation for the left mesh:

$$\frac{di_L(t)}{dt} = \frac{u_1(t)}{2L} - \frac{i_L(t)R}{2L} + \frac{1}{2L}\left[i_L(t)R - u_2(t) - 2v_c(t)\ \right]$$

Simplifying this equation:

$$\frac{di_L(t)}{dt} = -\frac{v_c(t)}{L} + \frac{i_L(t)R}{2L} + \frac{1}{2L}u_1(t) - \frac{1}{2L}u_2(t)$$

The output, y(t), is given as the voltage across the resistor, or equivalently:

$$y(t) = 2v_c(t) + u_2(t)$$

Putting our equations in the standard form:

$$\dot{x}_1 = -\frac{1}{L}x_2 + \frac{1}{2L}u_1(t) - \frac{1}{2L}u_2(t)$$

$$\dot{x}_2 = -\frac{1}{C}x_1 - \frac{2}{RC}x_2(t) - \frac{1}{RC}u_2(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{2}{RC} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2L} & -\frac{1}{2L} \\ 0 & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

which in matrix form is

$$y = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

(b) Now that you have finished (a), tell me how many state variables you should have used, and identify which voltages and/or currents they correspond to :)

Although there are four memory elements in this problem, and memory elements are conventionally chosen as state variables, only two of them should be chosen in this problem, and they correspond to the voltage in either capacitor and the current in either inductor. This simplification is borne out of the fact that the KVL relationship for each of the circuit's respective meshes gives that the significant variables associated with each memory element add, thus giving a "coupling" effect that gives rise to the need to use only two state variables.