Analytical Solution

The output of a system is given by the convolution of the input with the impulse response of the system.

$$x[n] = x_1 + x_2 + x_3 \ where \ x_1 = 1, \ x_2 = \sin\left(\frac{pi \ n}{4}\right), \ x_3 = \sin\left(\frac{pi \ n}{2}\right)$$

$$y[n] = y_1 + y_2 + y_3 \ where \ y_1 = x_1 * h, \ y_2 = x_2 * h, \ y_3 = x_3 * h$$

Owing to difficulties with calculating certain transforms, superposition was exploited to use different techniques for deriving the solution. Specifically, the z-transform was used to find y_1, and discrete time Fourier transforms for y_2 and y_3 .

Y1

Z transform derivation of h[n].

$$y[n-1]+(.9)y[n]=1.9x[n-1]$$
$$z \cdot Y[z]+(.9)Y[z]=1.9z \cdot X[z]$$
$$Y[z] \cdot (z+.9)=1.9z \cdot X[z]$$
$$Y\frac{[z]}{X}[z]=H[z]=\frac{1.9 \cdot z}{z+.9}$$

Z transform of X1

We operate on the assumption that x(n) = 0 for n<0; thus:

$$x_1=1=U(n)$$

$$X_1[z]=\frac{1}{1+\dfrac{1}{z}}$$

$$Y_1[z]=X_1[z]*H[z]$$

$$Y_1[z]=\frac{1}{1+\dfrac{1}{z}} \cdot 1.9 \cdot \frac{z}{z+.9}$$

$$Y_1[z]=1.9 \cdot \frac{z}{z-.1-\dfrac{.9}{z}}$$

$$Y_1[z]=1.9 \cdot \frac{z^2}{z^2-.1z-.9}$$

$$Y_1[z]=1.9 \cdot (1+\frac{.1z+.9}{z^2-.1z-.9})$$

$$Y_1[z]=1.9 \cdot (\frac{(1+\dfrac{1}{1.9})}{z-1}+\frac{\dfrac{.9}{1.9}-.9}{1+\dfrac{.9}{z}})$$

$$Y_1[z]=1.9 \cdot (delta(n)+\frac{1}{1.9}U(n)+\frac{.9}{1.9}-.9 \cdot (-.9)^n U(n))$$

Partial fractions constants

$$(z+.9)a+(z-1)b \equiv .1z+.9$$
$$z(a+b)+.9a-b \equiv .1z+.9$$

$$a+b=.1$$
$$.9a-b=.9$$
$$b=.9a-.9 \rightarrow a+(.9a-.9)=.1$$

$$a=\frac{1}{1.9}$$

$$b=\frac{.9}{1.9}-.9$$

Y2

For Y2 and Y3, the Fourier transform is used. The breakthrough comes from Section 5.5.1 of the book.

$$x_2 = \sin\left(\frac{pi\, n}{4}\right)$$

$$X_2(w) = \sum j \cdot pi \cdot \left[delta\left(w + \frac{pi}{4} - 2\, pi\, k\right) - delta\left(w - \frac{pi}{4} - 2\, pi\, k\right) \right]$$

This transform comes from the tables. The limits of the sum are k = -infinity to +infinity.

$$H(w) = \frac{1.9 \cdot 1}{1 + .9 \cdot e^{-jw}}$$

This comes from 5.45 a) worked successfully on my test.

$$Y_2(w) = H(w) X_2(w) = \sum H(w) \cdot j \cdot pi \cdot \left[delta\left(w + \frac{pi}{4} - 2 \cdot pi \cdot k\right) - delta\left(w - \frac{pi}{4} - 2 \cdot pi \cdot k\right) \right]$$

The first trick: Since H(w) is periodic, and is being sifted by the delta function, the value of H(w) is a constant, namely H(c), where c is the constant frequency shift of the delta function in question.

$$Y_2(w) = \sum j \cdot pi \cdot \left[H\left(\frac{pi}{4}\right) \cdot delta\left(w + \frac{pi}{4} - 2 \cdot pi \cdot k\right) - H\left(\frac{-pi}{4}\right) \cdot delta\left(w - \frac{pi}{4} - 2 \cdot pi \cdot k\right) \right]$$

The second trick: Represent H(c) as a phasor.

$$Y_2(w) = \sum j \cdot pi \cdot \left[\left|H\left(\frac{pi}{4}\right)\right| e^{-j < H\left(\frac{pi}{4}\right)} \cdot delta\left(w + \frac{pi}{4} - 2 \cdot pi \cdot k\right) - \left|H\left(\frac{-pi}{4}\right)\right| e^{-j < H\left(\frac{pi}{4}\right)} \cdot delta\left(w - \frac{pi}{4} - 2 \cdot pi \cdot k\right) \right]$$

$$y_2[n] = \left|H\left(\frac{pi}{4}\right)\right| \cdot \sin\left(\frac{pi}{4} \cdot n + angle\, H\left(\frac{pi}{4}\right)\right)$$

$$y_2[n] = 8.021 \cdot \sin\left(\frac{pi}{4} \cdot n + .3709\right)$$

Y3

Y3 works exactly the same as Y2, but with different constants.

$$x_3 = \sin\left(\frac{pi\, n}{2}\right)$$

$$X_3(w) = \sum j \cdot pi \cdot \left[delta\left(w + \frac{pi}{2} - 2\, pi\, k\right) - delta\left(w - \frac{pi}{2} - 2\, pi\, k\right) \right]$$

This transform comes from the tables. The limits of the sum are k = -infinity to +infinity.

$$H(w) = \frac{1.9 \cdot 1}{1 + .9 \cdot e^{-jw}}$$

This comes from 5.45 a) worked successfully on my test.

$$Y_3(w) = H(w)X_2(w) = \sum H(w) \cdot j \cdot pi \cdot \left[delta\left(w + \frac{pi}{2} - 2 \cdot pi \cdot k\right) - delta\left(w - \frac{pi}{2} - 2 \cdot pi \cdot k\right) \right]$$

The first trick: Since H(w) is periodic, and is being sifted by the delta function, the value of H(w) is a constant, namely H(c), where c is the constant frequency shift of the delta function in question.

$$Y_2(w) = \sum j \cdot pi \cdot \left[H\left(\frac{pi}{2}\right) \cdot delta\left(w + \frac{pi}{2} - 2 \cdot pi \cdot k\right) - H\left(\frac{-pi}{2}\right) \cdot delta\left(w - \frac{pi}{2} - 2 \cdot pi \cdot k\right) \right]$$

The second trick: Represent H(c) as a phasor.

$$Y_2(w) = \sum j \cdot pi \cdot \left[\left| H\left(\frac{pi}{2}\right) \right| e^{-j < H\left(\frac{pi}{2}\right)} \cdot delta\left(w + \frac{pi}{2} - 2 \cdot pi \cdot k\right) - \left| H\left(\frac{-pi}{2}\right) \right| e^{-j < H\left(\frac{pi}{2}\right)} \cdot delta\left(w - \frac{pi}{2} - 2 \cdot pi \cdot k\right) \right]$$

$$y_2[n] = \left| H\left(\frac{pi}{2}\right) \right| \cdot \sin\left(\frac{pi}{2} \cdot n + angle\, H\left(\frac{pi}{2}\right)\right)$$

$$y_2[n] = 1.4123 \cdot \sin\left(\frac{pi}{2} \cdot n + .7328\right)$$

Numerical Solution

The numerical solution numerically convolves the input and impulse responses using the filter command, and also directly calculates the output function determined from the analytical solution. The plot below shows the results.
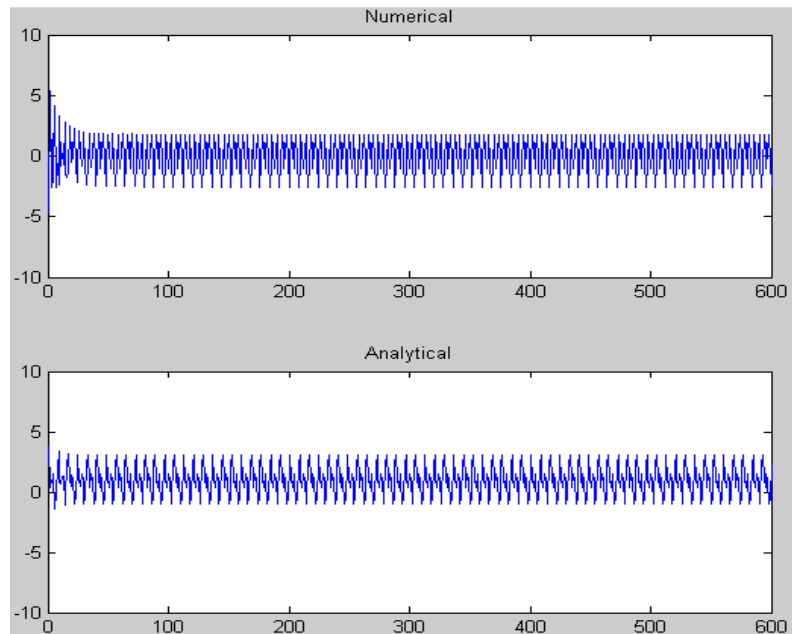
Better correspondence was obtained on earlier attempts, but Matlab has become uncooperative at the time of this writing. Matlab code appears below.

```matlab
clear all
clear variables
time=600;
x=zeros(1,time);
h=zeros(1,time);
out=zeros(1,time);
n=zeros(1,time);
a=ones(1,time);
n=1;

alpha = 1/1.9;%From partial fractions - they get messy
beta=.9/1.9-.9;
for k=1:time
    n=k;
    %To be convolved
    x(k)=1+sin(pi*n/4)+sin(pi*n/2);
    h(k)=1.9*heaviside(n)*(-.9)^n;
    %To be evaluated directly
    out(k)=1.9*(dirac(n)+alpha*heaviside(n)+beta*heaviside(n)*(-.9)^n)
+1.082*sin(pi*n/4 + .3709)+1.4123*sin(pi*n/2 + .7328);
end

y=filter(h,a,x);%More or less equivalent to discrete convolution for our purposes

subplot(2,1,1); plot(y);axis([0 time -10 10])
title('Numerical')
subplot(2,1,2); plot(out);axis([0 time -10 10])
title('Analytical')
```