Fourier Analysis of Discrete-Time Systems

Gary W. Weasel, Jr (gww44@msstate.edu)
ECE 3163, Section 01

Problem Procedure

The problem 5.45 from page 277-278 says to consider the discrete-time system given by the input/output difference equation:

$$y[n+1]+0.9y[n]=1.9x[n+1] \tag{1.1}$$

From part (a) and part (b) of the problem, we are able to prove that the impulse response satisfies equation (1.1). Using part (b), we compute the Discrete-Time Fourier Transform (DTFT) of the impulse response proven in part (a). This impulse response and its DTFT are as follows:

$$h[n]=1.9(-0.9)^n u[n] \tag{1.2}$$

$$H(\Omega)=\frac{1.9e^{j\Omega}}{e^{j\Omega}+0.9} \tag{1.3}$$

With this information, part (c) of problem 5.45 then asks to compute the output response *y[n]* to an input of the following:

$$x[n]=1+\sin(\frac{\pi n}{4})+\sin(\frac{\pi n}{2}) \tag{1.4}$$

Since the input provided is linear, in can be split into three separate parts and then computed separately. So now the three equations to work with are seen below:

$$x_1[n]=1 \tag{1.5}$$

$$x_2[n]=\sin(\frac{\pi n}{4}) \tag{1.6}$$

$$x_3[n]=\sin(\frac{\pi n}{2}) \tag{1.7}$$

According to theory, the output response *y[n]* resulting from the application of input *x[n]* can be determined using the following equation:

$$y[n]=h[n]*x[n] \leftrightarrow Y(\Omega)=H(\Omega)X(\Omega) \tag{1.8}$$

Since we have equation (1.3), we can compute the output response if we obtain *X(Ω)* and apply it to equation (1.8). Therefore, the DTFT will need to be applied to all three components (1.5, 1.6, and 1.7) of the original input from (1.4). Once *X(Ω)* for each component has been obtained, it and *H(Ω)* are plugged into equation (1.8), *H(Ω)* is applied to the input of the system to generate the output. After *H(Ω)* has been applied to the system, the inverse DTFT is performed to obtain *y[n]*. The work for each component is shown below.

However, there are several essential equations required during the computation of each components. In order to compute $H(\Omega)$ and actually proceed in the computation, the following equation:

$$H(\Omega)\delta(\Omega + c) = H(-c)\delta(\Omega + c) \tag{1.9}$$

In these examples, there will consistently exist a 2πk in the c variable. This 2πk can be disregarded due to the fact that $H(\Omega)$ is periodic with period 2π. So it can be observed that:

$$H(-\Omega_O + 2\pi k) = H(-\Omega_O) \tag{1.10}$$

Once an numerical value exists within the impulse response function, the value can either be applied straight to the DTFT equation of the impulse response function or the following equation can be used in respect to the magnitude and angle of the impulse response:

$$\begin{aligned} H(\Omega_O) &= \left|H(\Omega_O)\right| \cdot e^{j\angle H(\Omega_O)} \\ H(-\Omega_O) &= \left|H(\Omega_O)\right| \cdot e^{-j\angle H(\Omega_O)} \end{aligned} \tag{1.11}$$

The transforms used in computing the output from each input component are as follows:

$$1, -\infty < t < \infty \leftrightarrow 2\pi\delta(\Omega - 2\pi k) \tag{1.12}$$

$$\sin(\Omega_O n + \Theta) \leftrightarrow \sum_{k=-\infty}^{\infty} j\pi[e^{-j\Theta}\delta(\Omega + \Omega_O - 2\pi k) - e^{j\Theta}\delta(\Omega - \Omega_O - 2\pi k)] \tag{1.13}$$

The following is the computation from $x_1[n]$ to $y_1[n]$. At the beginning the transform from equation (1.12) is used to transform $x_1[n]$.

$$x_1[n] = 1$$

$$X_1(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi \cdot \delta(\Omega - 2\pi k)$$

$$Y_1(\Omega) = X_1(\Omega) \cdot H(\Omega)$$

$$Y_1(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi \cdot H(\Omega) \cdot \delta(\Omega - 2\pi k)$$

From this point in the computation, we use equations (1.9, 1.10) to determine that in this case from computing the first part that $H(\Omega) = H(0)$.

$$Y_1(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi \cdot H(0) \cdot \delta(\Omega - 2\pi k)$$

$$Y_1(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi \cdot \delta(\Omega - 2\pi k)$$

$$y_1[n] = 1$$

Next the second component $x_2[n]$ is computed the same way except with a different transform (from equation 1.13), and an additional equation:

$$x_2[n] = \sin(\frac{\pi n}{4})$$

$$X_2(\Omega) = \sum_{k=-\infty}^{\infty} j\pi[\delta(\Omega + \tfrac{\pi}{4} - 2\pi k) - \delta(\Omega - \tfrac{\pi}{4} - 2\pi k)]$$

$$Y_2(\Omega) = X_2(\Omega)H(\Omega)$$

$$Y_2(\Omega) = \sum_{k=-\infty}^{\infty} j\pi[H(\Omega)\delta(\Omega + \tfrac{\pi}{4} - 2\pi k) - H(\Omega)\delta(\Omega - \tfrac{\pi}{4} - 2\pi k)]$$

$$Y_2(\Omega) = \sum_{k=-\infty}^{\infty} j\pi[H(-\tfrac{\pi}{4})\delta(\Omega + \tfrac{\pi}{4} - 2\pi k) - H(\tfrac{\pi}{4})\delta(\Omega - \tfrac{\pi}{4} - 2\pi k)]$$

In order to compute the impulse response, equation (1.11) allows the expansion of *H(Ω)* into a exponent form:

$$Y_2(\Omega) = \sum_{k=-\infty}^{\infty} j\pi[\left|H(\tfrac{\pi}{4})\right|e^{-j\angle H(\frac{\pi}{4})}\delta(\Omega + \tfrac{\pi}{4} - 2\pi k) - \left|H(\tfrac{\pi}{4})\right|e^{j\angle H(\frac{\pi}{4})}\delta(\Omega - \tfrac{\pi}{4} - 2\pi k)]$$

$$y_2[n] = \left|H(\tfrac{\pi}{4})\right|\sin(\frac{\pi}{4}n + \angle H(\frac{\pi}{4}))$$

$$y_2[n] = 1.08 \cdot \sin(\frac{\pi}{4}n + 0.37)$$

Finally, the final computation for $x_3[n]$ is as follows:

$$x_3[n] = \sin(\frac{\pi n}{2})$$

$$X_3(\Omega) = \sum_{k=-\infty}^{\infty} j\pi[\delta(\Omega + \tfrac{\pi}{2} - 2\pi k) - \delta(\Omega - \tfrac{\pi}{2} - 2\pi k)]$$

$$Y_3(\Omega) = X_3(\Omega)H(\Omega)$$

$$Y_3(\Omega) = \sum_{k=-\infty}^{\infty} j\pi[H(\Omega)\delta(\Omega + \tfrac{\pi}{2} - 2\pi k) - H(\Omega)\delta(\Omega - \tfrac{\pi}{2} - 2\pi k)]$$

$$Y_3(\Omega) = \sum_{k=-\infty}^{\infty} j\pi[H(-\tfrac{\pi}{2})\delta(\Omega + \tfrac{\pi}{2} - 2\pi k) - H(\tfrac{\pi}{2})\delta(\Omega - \tfrac{\pi}{2} - 2\pi k)]$$

$$Y_3(\Omega) = \sum_{k=-\infty}^{\infty} j\pi[|H(\tfrac{\pi}{2})|e^{-j\angle H(\frac{\pi}{2})}\delta(\Omega + \tfrac{\pi}{2} - 2\pi k) - |H(\tfrac{\pi}{2})|e^{j\angle H(\frac{\pi}{2})}\delta(\Omega - \tfrac{\pi}{2} - 2\pi k)]$$

$$y_3[n] = |H(\tfrac{\pi}{2})|\sin(\frac{\pi}{2}n + \angle H(\frac{\pi}{2}))$$

$$y_3[n] = 1.41 \cdot \sin(\frac{\pi}{2}n + 0.73)$$

After computing all of the output components, all three of them can be computed to create the overall output response from the function, therefore generating the equation as seen below:

$$y[n] = 1 + 1.08 \cdot \sin(\frac{\pi}{4}n + 0.37) + 1.41 \cdot \sin(\frac{\pi}{2}n + 0.73) \tag{1.14}$$

MATLAB Simulation

In order to verify the solution obtained from the previous section, a MATLAB simulation should be run. The method established for verifying the solution is to compute the convolution of the input signal and the impulse response signal using MATLAB, and then graphing equation (1.14) and checking if they overlap.

The following is the block of code used to perform the convolution and graph both sets of data.

```
% problem.m - Signals & Systems Problem 5.45c
% Last modified: March 27, 2009

% Set up the data range for n and compute x[n]
n=0:1:199;
x=1+sin(pi/4*n)+sin(pi/2*n);

% Run the data range for k and compute h[k]
for k=1:200;
    h(k)=1.9*(-0.9)^(k-1);
end

% Perform x[n]*h[k] (convolution) and store it to z.  Also,
% store equation 1.14 to y.
z=conv(x,h);
y=1+1.08*sin(pi/4*n+0.37)+1.41*sin(pi/2*n+0.73);

% Graph y in red and z with dashed lines and square points
stem(y,'r');
hold
stem(z,'--s');
```

Figure 1 – MATLAB Code to Simulate the output response

One important mention of the results for graphing the convolution of the two signals is that since the convolution is not done for infinity samples, the graph eventually decays to 0. Would we be able to graph an infinite convolution, the decay at the end of the graph would not exist.

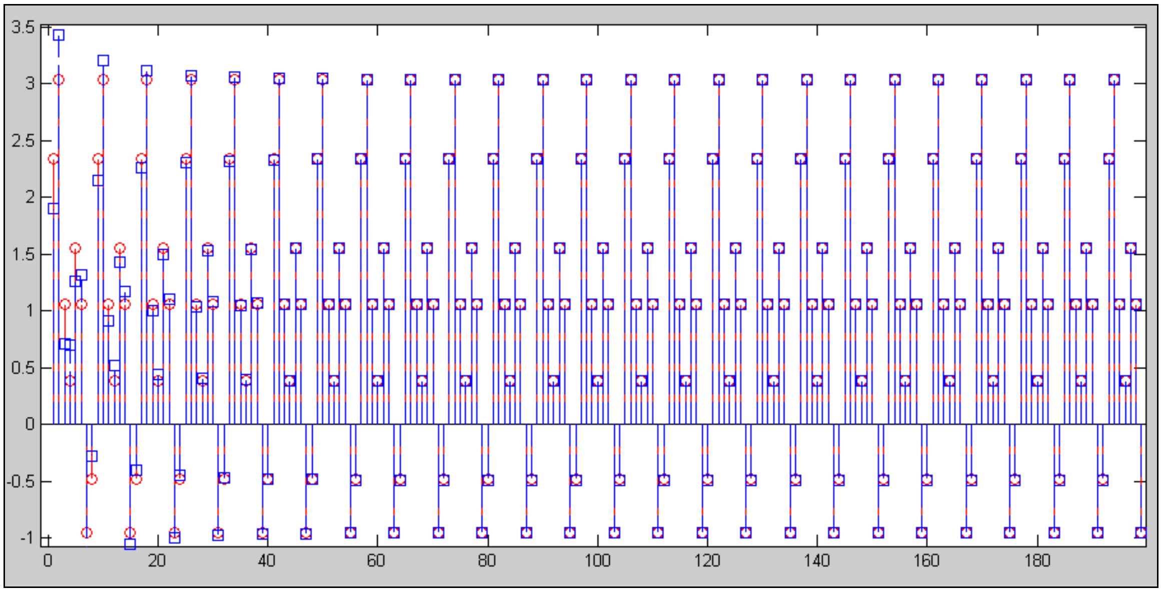

Figure 2 – Simulation Results for both the convolution and equation (1.14)

Figure 2 above validates that equation (1.14) does match the simulation performed by MATLAB. The two graphs are laid over each other, with the calculated convolution being represented by blue dashed lines with squares for plot points, and equation (1.14) being indicated in red and with circles for plot points.

At the beginning of the dataset there are some slot variations between the simulation and the plotted equation, but these variations quickly smooth out.