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5.45) c) $y[n+1] + .9\, y[n] = 1.9\, x[n+1]$ $\qquad x[n] = 1 + \sin\left(\frac{\pi n}{4}\right) + \sin\left(\frac{\pi n}{2}\right)$

from a) + b) $h[n] \leftrightarrow H(\omega)$

$$1.9(-.9)^n u[n] \leftrightarrow \frac{1.9e^{j\omega}}{e^{j\omega}+.9} \quad \text{using DTFT}$$

$X_1[n] = 1$ using DTFT

$$\boxed{X_1(\omega) = \sum_{K=-\infty}^{\infty} 2\pi\, \delta(\omega - 2\pi K)}$$

$$H(0) = \frac{1.9e^{j(0)}}{e^{j(0)}+.9} = \frac{1.9}{1+.9} = 1$$

$$1 \leftrightarrow \sum_{K=-\infty}^{\infty} 2\pi\, \delta(\omega - 2\pi k) \longrightarrow$$

$Y_1(\omega) = X(\omega) H(\omega)$

$$\sum_{K=-\infty}^{\infty} 2\pi\, H(\omega)\, \delta(\omega - 2\pi k)$$

$$\sum_{K=-\infty}^{\infty} 2\pi\, H(0)\, \delta(\omega - 2\pi k)$$

$$\sum_{K=-\infty}^{\infty} 2\pi\, (1)\, \delta(\omega - 2\pi k)$$

$$\boxed{y_1[n] = 1}$$

$X_2[n] = \sin\left(\frac{\pi n}{4}\right)$ $\qquad \omega_0 = \pi/4$

$$\sin(\omega_0 n) \leftrightarrow \sum_{K=-\infty}^{\infty} j\pi\left[\delta(\omega + \omega_0 - 2\pi K) - \delta(\omega - \omega_0 - 2\pi K)\right]$$

$$\boxed{X_2(\omega) = \sum_{K=-\infty}^{\infty} j\pi\left[\delta(\omega + \pi/4 - 2\pi K) - \delta(\omega - \pi/4 - 2\pi K)\right]}$$

$\omega_0 = \pi/4 \qquad \theta = \angle H(\pi/4)$

$Y_2(\omega) = X_2(\omega) H(\omega)$

$$\sum_{K=-\infty}^{\infty} j\pi\left[H(\omega)\,\delta(\omega + \pi/4 - 2\pi K) - H(\omega)\,\delta(\omega - \pi/4 - 2\pi K)\right]$$

$$\sum_{K=-\infty}^{\infty} j\pi\left[H(-\pi/4)\,\delta(\omega + \pi/4 - 2\pi K) - H(\pi/4)\,\delta(\omega - \pi/4 - 2\pi K)\right]$$

$$\sum_{K=-\infty}^{\infty} j\pi\left[\left|H(-\pi/4)\right| e^{-j\angle H(\pi/4)}\,\delta(\omega + \pi/4 - 2\pi K) - \left|H(\pi/4)\right| e^{j\angle H(\pi/4)}\,\delta(\omega - \pi/4 - 2\pi K)\right]$$

$$y_2[n] = \left|H(-\pi/4)\right| \sin\left(\pi/4\, n + \angle H(\pi/4)\right)$$

$$\boxed{y_2[n] = 1.08214 \sin(\pi/4\, n + .370902)}$$

5.45 c) Cont

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$$\boxed{x_3[n] = \sin\left(\frac{\pi n}{2}\right)} \qquad \theta = \angle H\left(\frac{\pi}{2}\right) \quad \omega_0 = \frac{\pi}{2}$$

$$\boxed{X_3(\omega) = \sum_{k=-\infty}^{\infty} j\pi\left[\delta\left(\omega+\frac{\pi}{2}-2k\pi\right)-\delta\left(\omega-\frac{\pi}{2}-2k\pi\right)\right]}$$

$$Y_3(\omega) = X_3(\omega)H(\omega)$$

$$\sum_{k=-\infty}^{\infty} j\pi\left[H(\omega)\,\delta\left(\omega+\frac{\pi}{2}-2k\pi\right)-H(\omega)\,\delta\left(\omega-\frac{\pi}{2}-2k\pi\right)\right]$$

$$\sum_{k=-\infty}^{\infty} j\pi\left[H\left(\frac{\pi}{2}\right)\delta\left(\omega+\frac{\pi}{2}-2k\pi\right)-H\left(\frac{\pi}{2}\right)\delta\left(\omega-\frac{\pi}{2}-2k\pi\right)\right]$$

$$\sum_{k=-\infty}^{\infty} j\pi\left[\left|H\left(\frac{\pi}{2}\right)\right|e^{-j\angle H\left(\frac{\pi}{2}\right)}\delta\left(\omega+\frac{\pi}{2}-2k\pi\right)-\left|H\left(\frac{\pi}{2}\right)\right|e^{j\angle H\left(\frac{\pi}{2}\right)}\delta\left(\omega-\frac{\pi}{2}-2k\pi\right)\right]$$

$$y_3[n] = \left|H\left(\frac{\pi}{2}\right)\right|\sin\left(\frac{\pi}{2}n+\angle H\left(\frac{\pi}{2}\right)\right)$$

$$\boxed{y_3[n] = 1.41421\sin\left(\frac{\pi}{2}n+.732815\right)}$$

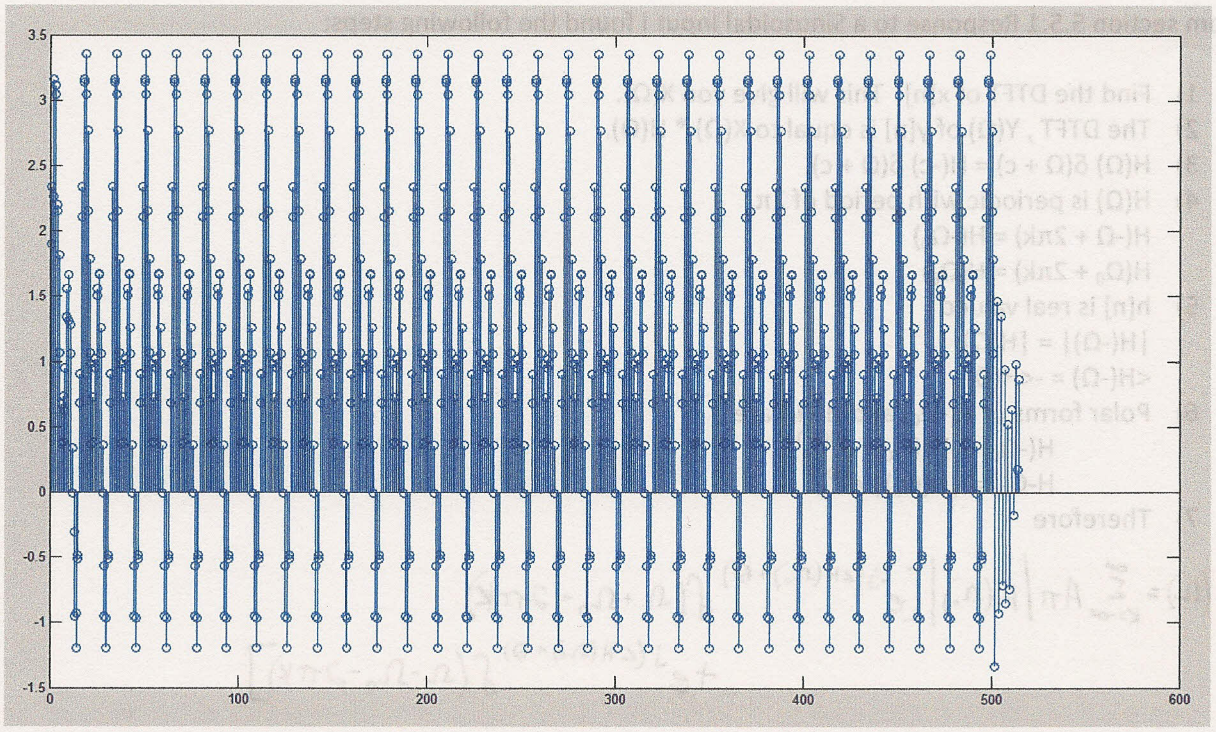
$$y[n] = y_1[n] + y_2[n] + y_3[n]$$

$$\boxed{y[n] = 1 + 1.08214\sin\left(\frac{\pi}{4}n+.370902\right)+1.41421\sin\left(\frac{\pi}{2}n+.732815\right)}$$

MATLAB Verification

Below is the MATLAB code and output that shows the analytical solution lines up with the simulation of the problem.

```
>> n = 0:0.5:250;
>> x = 1 + sin(pi*n/4) + sin(pi*n/2);
>> for i = 1:15;
h(i) = 1.9 * ( -.9)^(i-1);
end;
>> z = conv(x,h);
>> y = 1 + (1.08214 * sin(pi*n/4 + .370902)) + (1.41421 * sin(pi*n/2 +.732815));
>> stem(z)
>> hold
Current plot held
>> stem(y)
```



Solution Description

To solve this problem I broke x[n] into three sub-equations, $x_1[n]$, $x_2[n]$, and $x_3[n]$. Once this was done I used the frequency domain method on each of these sub-equations.

1) Compute the transform of the impulse response.
2) Compute the transform of the input.
3) Multiply.
4) Compute the inverse transform.

Once this method was used I added my three sub-equations back together to get the result. Listed below is a more detailed description of this process as well as properties and equations that I used to calculate the analytical solution.

From section 5.5.1 Response to a Sinusoidal Input I found the following steps:

1) Find the DTFT of x[n]. This will give you $X(\Omega)$.
2) The DTFT , $Y(\Omega)$ of y[n] is equal to $X(\Omega) * H(\Omega)$.
3) $H(\Omega)\,\delta(\Omega + c) = H(-c)\,\delta(\Omega + c)$.
4) $H(\Omega)$ is periodic with period of 2π
   $H(-\Omega + 2\pi k) = H(-\Omega_0)$
   $H(\Omega_0 + 2\pi k) = H(\Omega_0)$
5) h[n] is real valued
   $|H(-\Omega)| = |H(\Omega)|$
   $<H(-\Omega) = -<H(\Omega)$
6) Polar forms of $H(-\Omega_0)$ and $H(\Omega_0)$ are
   $H(-\Omega_0) = |H(\Omega_0)|\, e^{-j(\Omega_0)}$
   $H-\Omega_0) = |H(\Omega_0)|\, e^{j(\Omega_0)}$
7) Therefore

$$Y(\Omega) = \sum_{k=-\infty}^{\infty} A\pi |H(\Omega_0)| \Big[e^{-j(\angle H(\Omega_0) + \theta)}\, \delta(\Omega + \Omega_0 - 2\pi k) + e^{j(\angle H(\Omega_0)+\theta)}\, \delta(\Omega - \Omega_0 - 2\pi k) \Big]$$

8) Taking the Inverse DTFT

$$y[n] = A|H(\Omega_0)| \cos(\Omega_0 n + \theta + \angle H(\Omega_0)), \quad n = 0, \pm 1, \pm 2, \ldots$$

Other formulas:

DTFT Pairs

$$1, \text{ all } n \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega - 2\pi k)$$

$$\sin(\omega_0 n) \leftrightarrow \sum_{k=-\infty}^{\infty} j\pi\left[\delta(\omega + \omega_0 - 2\pi k) - \delta(\omega - \omega_0 - 2\pi k)\right]$$

$$\cos\theta + i\sin\theta = e^{i\theta}$$