Name: ____

Problem	Points	Score
1	35	
2	30	
3	35	
Total	100	

Notes:

(1) The exam is closed books and notes except for one double-sided sheet of notes.

(2) Please indicate clearly your answer to the problem.

(3) If I can't read or follow your solution, it is wrong and no partial credit will be awarded.

Problem No. 1:

8. **The Z channel.** The Z-channel has binary input and output alphabets and transition probabilities $p(y|x)$ given by the following matrix:

$$Q = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} \qquad x, y \in \{0, 1\}$$

Find the capacity of the Z-channel and the maximizing input probability distribution.

Solution: *The Z channel.* First we express $I(X;Y)$, the mutual information between the input and output of the Z-channel, as a function of $x = \Pr(X = 1)$:

$$\begin{aligned} H(Y|X) &= \Pr(X = 0) \cdot 0 + \Pr(X = 1) \cdot 1 = x \\ H(Y) &= H(\Pr(Y = 1)) = H(x/2) \\ I(X;Y) &= H(Y) - H(Y|X) = H(x/2) - x \end{aligned}$$

Since $I(X;Y) = 0$ when $x = 0$ and $x = 1$, the maximum mutual information is obtained for some value of x such that $0 < x < 1$.

Using elementary calculus, we determine that

$$\frac{d}{dx} I(X;Y) = \frac{1}{2} \log_2 \frac{1 - x/2}{x/2} - 1,$$

which is equal to zero for $x = 2/5$. (It is reasonable that $\Pr(X = 1) < 1/2$ because $X = 1$ is the noisy input to the channel.) So the capacity of the Z-channel in bits is $H(1/5) - 2/5 = 0.722 - 0.4 = 0.322$.

Problem No. 2:

4. **Quantized random variables.** Roughly how many bits are required on the average to describe to 3 digit accuracy the decay time (in years) of a radium atom if the half-life of radium is 80 years? Note that half-life is the median of the distribution.

   Solution: *Quantized random variables.* The differential entropy of an exponentially distributed random variable with mean $1/\lambda$ is $\log \frac{e}{\lambda}$ bits. If the median is 80 years, then

$$\int_0^{80} \lambda e^{-\lambda x}\, dx = \frac{1}{2} \tag{8.11}$$

   or

$$\lambda = \frac{\ln 2}{80} = 0.00866 \tag{8.12}$$

   and the differential entropy is $\log e/\lambda$. To represent the random variable to 3 digits $\approx$ 10 bits accuracy would need $\log e/\lambda + 10$ bits $= 18.3$ bits.

Problem No. 3:

4. **Exponential noise channels.** Consider an additive noise channel $Y_i = X_i + Z_i$, where Z_i is i.i.d. exponentially distributed noise with mean μ. Assume that we have a mean constraint on the signal, i.e., $EX_i \leq \lambda$. Show that the capacity of such a channel is $C = \log(1 + \frac{\lambda}{\mu})$.

Solution: *Exponential noise channels*

Just as for the Gaussian channel, we can write

$$\begin{aligned}
C &= \max_{f(X):EX\leq\lambda} I(X;Y) && (9.30)\\
&= \max_{f(X):EX\leq\lambda} h(Y) - h(Y|X) && (9.31)\\
&= \max_{f(X):EX\leq\lambda} h(Y) - h(Z|X) && (9.32)\\
&= \max_{f(X):EX\leq\lambda} h(Y) - h(Z) && (9.33)\\
&= \max_{f(X):EX\leq\lambda} h(Y) - (1 + \ln\mu) && (9.34)\\
& && (9.35)
\end{aligned}$$

Now $Y = X + Z$, and $EY = EX + EZ \leq \lambda + \mu$. Given a mean constraint, the entropy is maximized by the exponential distribution, and therefore

$$\max_{EY\leq\lambda+\mu} h(Y) = 1 + \ln(\lambda+\mu) \qquad (9.36)$$

Unlike normal distributions, though, the sum of two exponentially distributed variables is not exponential, so we cannot set X to be an exponential distribution to achive the right distribution of Y. Instead, we can use characterstic functions to find the distribution of X. The characteristic function of an exponential distribution

$$\psi(t) = \int \frac{1}{\mu} e^{-\frac{x}{\mu}} e^{-itx}\, dx = \frac{1}{1 - i\mu t} \qquad (9.37)$$

The distribution of X that when added to Z will give an exponential distribution for Y is the ratio of the characterstic functions

$$\begin{aligned}
\psi_X(t) &= \frac{1 - i\mu t}{1 - i(\lambda+\mu)t} && (9.38)\\
& && (9.39)
\end{aligned}$$

which can been seen to correpond to mixture of a point mass and an exponential distribution. If

$$X = \begin{cases} 0, & \text{with probability } \frac{\mu}{\lambda+\mu} \\ X_e, & \text{with probability} \frac{\lambda}{\lambda+\mu} \end{cases} \qquad (9.40)$$

where X_e has an exponential distribution with parameter $\mu + \lambda$, we can verify that the characterstic function of X is correct.

Using the value of entropy for exponential distributions, we get

$$C = h(Y) - h(Z) = 1 + + \ln(\lambda+\mu) - (1 + \ln\mu) = \ln\left(1 + \frac{\lambda}{\mu}\right) \qquad (9.41)$$