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Entropy of Continuous Gussian Random Variable

<u>You can change the values in this MATLAB Live Script file to play with different values and see the results. Checked on MATLAB 2016b.</u>

Gaussian Random Variable

The probability density function $p(x)$ of Gaussian Random Variable is as follows:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where μ and σ are *mean* and *standard deviation* of the Gaussian random variable respectively. Let us consider $\mu = 0$ and $\sigma = 1$ for simplicity (however, we can change them in this code to play with results).

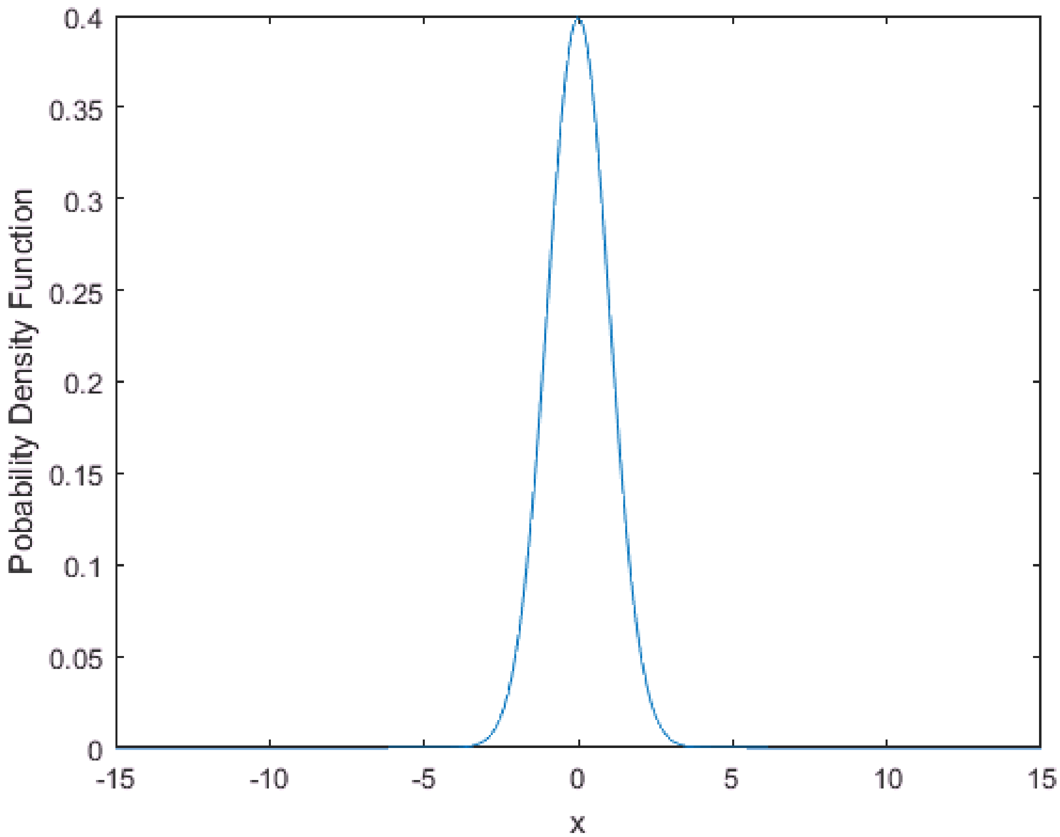
```
syms x
sigma = 1; %% standard deviation
miu = 0; %% mean
px = 1/sqrt(2*pi*sigma)*exp(-(x-miu)^2/(2*sigma^2))
```

px =

$$\frac{7186705221432913\, e^{-\frac{x^2}{2}}}{18014398509481984}$$

Above is the short form of the density function in hand. We can visualize this density function as:

```
newplot; fplot(px);
xlabel('x'); ylabel('Pobability Density Function')
axis([-15 15 -inf inf])
```



Lets ensure that our PDF is correct. Integrate it over all the values of x to get the unity answer.

```
pdffun = @(x) (1/sqrt(2*pi*sigma)*exp(-(x-miu).^2/(2*sigma^2)));  % Gaussian PDF
ensure = integral(pdffun,-inf,inf) %% ensuring that the function is correct
```

```
ensure = 1.0000
```

Thus, our PDF is correct.

Entropy of Continuous Gaussian Random Variable calculated from $-\infty$ to ∞

Entropy is given by:

$$H(X) = -\int_{-\infty}^{\infty} p(x)\log_2(p(x))\,\mathrm{dx}$$

Lets calculate it:

```
entropyfun = @(x) (-1/sqrt(2*pi*sigma)*exp(-(x-miu).^2/(2*sigma^2)))...
    .*log2((1/sqrt(2*pi*sigma)*exp(-(x-miu).^2/(2*sigma^2))));  ...
    % entropy function of Gaussian PDF
entropy = integral(entropyfun,-inf,inf) %% calculating entropy for Gaussian PDF
```

```
Warning: Infinite or Not-a-Number value encountered.
entropy = NaN
```

Above result clearly shows that entropy is infinite if the integration is done for all the possible values of x.

Entropy of Continuous Gaussian Random Variable calculated for a finite range of X

Lets restrict x from -10 to 10.

```
entropy = integral(entropyfun,-10,10) %% calculating entropy for Gaussian PDF
```

```
entropy = 2.0471
```

Note that the entropy above has decreased to 2.0471 bits. This is because, now our randomness (value of x) has decreased from $-\infty \longrightarrow \infty$ to $10 \longrightarrow 10$.

Lets restrict x from -3 to 3.

```
entropy = integral(entropyfun,-3,3) %% calculating entropy for Gaussian PDF
```

```
entropy = 2.0224
```

Note that the entropy has decreased because the randomness is less from $-3 \rightarrow 3$ compared to the randomness from $-10 \rightarrow 10$; however, we will still need at least 3 bits to express these two random variables (remember, $H(X) \le \text{bits required} \le 1 + H(X)$.

Lets introduce an interesting result now. Clculate entropy such that x is taken from -100 to 100.

```
entropy = integral(entropyfun,-100,100) %% calculating entropy for Gaussian PDF
```

```
Warning: Infinite or Not-a-Number value encountered.
entropy = NaN
```

The question is, why does above result approache to infinity? The answer is '*if the range of x is from -100 to 100, the entropy exceeds the maximum value supported by MATLAB*.'

```
clc; clear all;
```

Comparison of Entropy for Gaussian and Uniform Distributions

For Continuous Case

The probability density function $p(x)$ of Gaussian Random Variable is as follows:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where μ and σ are mean and standard deviation of the Gaussian random variable respectively. Let us consider $\mu = 0$ and $\sigma = 1$ for simplicity.

The probability density function $q(x)$ of Uniform Random Variable is as follows:

$$q(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

where a and b are the bounderies of the probability density function.

```
syms x b a
sigma = 1; %% standard deviation
miu = 0; %% mean
px = 1/sqrt(2*pi*sigma)*exp(-(x-miu)^2/(2*sigma^2))
```

px =

$$\frac{7186705221432913\, e^{-\frac{x^2}{2}}}{18014398509481984}$$

```
qx = 1/(b-a)
```

qx =

$$-\frac{1}{a-b}$$

Entropy for the above mentioned random variables is given by:

$$H_{\text{Gaussian}} = -\int_{-\infty}^{\infty} p(x)\log_2(p(x))\,\mathrm{dx}$$

$$H_{\text{Uniform}} = -\int_{-\infty}^{\infty} q(x)\log_2(q(x))\,\mathrm{dx}$$

Lets calculate entropy using above-mentioned forumulas for x ranging from -2 to 2.

```
a = -2;b = 2; %% range for calculating entropy
entropy_function_gaussian = @(x) ...
    (-1/sqrt(2*pi*sigma)*exp(-(x-miu).^2/(2*sigma^2))) .* ...
    log2((1/sqrt(2*pi*sigma)*exp(-(x-miu).^2/(2*sigma^2))));  ...
    % entropy function of Gaussian PDF
entropy_Gaussian = integral(entropy_function_gaussian,a,b) ...
    %% calculating entropy for Gaussian PDF
```

```
entropy_Gaussian = 1.7982
```

```
uniform_f = @(x) (-(1./(b-a)).*(x.^0)) .* log2(((1./(b-a)).*(x.^0)));  ...
    % entropy function for Uniform PDF
entropy_unifrom = integral(uniform_f,a,b) ...
    %% calculating entropy for Uniform PDF
```

```
entropy_unifrom = 2
```

Entropy (uncertainity) of Gaussian random variable is less than the entropy of uniform random variable because Gaussian random variable has more predictability near its mean.

Proof for discrete case is similar and has been left due to time constraint.