question_3

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Theorem 8.3.1

If the density f(x) of the random variable X is Riemann integrable, then

$H(X^{\Delta}) + \log \Delta \to h(f) = h(X)$ as $\Delta \to 0$

Thus, the entropy of an n-bit quantization of a continuous random variable X is approximately h(X) + n.

Question

Suppose Z = X + Y. What is the equivalent of Theorem 8.3.1?

Answer

$F_Z(z) = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{z-x} f(x, y) dy dx$

The pdf of Z is found using Leibniz integration:

$$\begin{aligned} f_Z(z) &= \frac{dF_Z(z)}{dz} \\ &= \int\limits_{-\infty}^{\infty} f_{X,Y}(z - y, y) dy \end{aligned}$$

(if X and Y were guaranteed independent, we could save ourselves some breath here by by just using convolution)

Z^Δ is a quantized random variable as described in the text.

The probability that $Z^\Delta = z_i$ is

$$\begin{aligned} p_i &= \int_{i\Delta}^{(i+1)\Delta} f_{X,Y}(z-y,y)dy \\ &= f_{X,Y}(z-y,y)_i\Delta \qquad \text{(I not sure of the best notation for this)} \end{aligned}$$

The entropy of the quantized version is

$$\begin{aligned} H(Z^\Delta) &= \sum_{-\infty}^{\infty} p_i \log \frac{1}{p_i} \\ &= \sum_{-\infty}^{\infty} f_{X,Y}(z-y,y)\Delta \log \frac{1}{f_{X,Y}(z-y,y)\Delta} \\ &= \sum f_{X,Y}(z-y,y)\Delta \log \frac{1}{f_{X,Y}(z-y,y)} + \sum f_{X,Y}(z-y,y)\Delta \log \frac{1}{\Delta} \\ &= \sum f_{X,Y}(z-y,y)\Delta \log \frac{1}{f_{X,Y}(z-y,y)} + \log \frac{1}{\Delta} \end{aligned}$$

We can do this final step because $\sum f_{X,Y}(z-y,y)\Delta = \int f_Z(z) = 1$

Similarly to the text's single dimensional example, if $f_{X,Y}(z-y,y)\Delta \log \frac{1}{f_{X,Y}(z-y,y)}$ is Riemann integrable, then the first term approaches the integral definition of the pdf of Z.

$$\begin{aligned} H(Z^\Delta) + \log \Delta \to h(f) = h(Z) \text{ as } \Delta \to 0 \\ H(X) + H(X|Y) + \log \Delta \to h(f) = h(Z) \text{ as } \Delta \to 0 \end{aligned}$$

Finally, we can break this up via the chain rule. The chain rule holds in the continuous case (Theorem 8.6.2 in text).

I had a bunch of stuff in here regarding a two-dimensional mean value theorem, but I realized I was adding unnecessary complexity, so I removed it.